

Textbooks for Tomorrow's Scientists

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Structural Analysis

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Brief Introduction to the Book

Structural Analysis is a basic course for undergraduate students with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and is also an introductory course for undergraduates to learn and master the analysis and design of beam, truss, frame, arch and composite structures for buildings, bridges and flight vehicles and so on. This textbook includes eight chapters and covers an introduction, kinematic analysis of plane member systems, analysis of statically determinate structures, the principle of virtual work and deflection calculation, force method, displacement method, influence lines of structures under moving loads, and matrix displacement method.

The present book emphasizes the fundamental theories, concepts, computational methods, and engineering applications of structural analysis. It can be used as a textbook for undergraduates with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and also is a helpful reference for associated engineers and professionals.

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Preface

Structural Analysis is a basic course for undergraduate students with majors in civil engineering, engineering mechanics, flight vehicle design, mechanical engineering, naval architecture, ocean engineering, etc., and is also an introductory course for undergraduates to learn and master the analysis and design of various beam, truss, frame, arch and composite structures for buildings, bridges and flight vehicles and so on. This textbook includes eight chapters and covers an introduction (chapter 1), kinematic analysis of plane member systems (chapter 2), analysis of statically determinate structures (chapter 3), principle of virtual work and deflection calculation (chapter 4), force method (chapter 5), displacement method (chapter 6), influence lines of structures under moving loads (chapter 7), and matrix displacement method (chapter 8). Chapters 5, 6, and 8 focus on addressing the analysis of statically indeterminate structures.

The present textbook emphasizes the fundamental theories, concepts, computational methods, and engineering applications of structural analysis. During the preparation of this book, we referred to many excellent textbooks and made efforts to renew some contents and reflect the time trends of digitalization and informatization. The main features of this textbook lie in the following aspects: (1) strengthened the interestingness and readability, and increased the brief introduction on the developmental history of structural analysis and the important figures; (2) substituted the kinematic method to construct exactly and rapidly the influence lines of forces of statically indeterminate structures proposed by the authors for the corresponding old knowledge, and highlighted the energy principle and method; (3) increased introducing the backgrounds of engineering applications for some theories and methods of structural analysis; (4) from the viewpoints of history, methodology, aesthetic appreciation, and creative thinking, inspected the structural analysis and strived to integrate the knowledge, capability and quality education to cultivate the innovative talents.

The main contributors to the contents of this book are Dixiong Yang (chapters 1 and 7), Junfeng Gu (chapters 2 and 3), Lei Yang (chapters 4 and 5), Jingjie Chen (chapter 6), and Xiaofei Hu (chapter 8).

During the process of writing this textbook, Professors Gengdong Cheng and Chunqiu Yang from the Dalian University of Technology, Professor Junfa Zhang from the Xi'an University of Technology, and Professor Jianbing Chen from Tongji University, China gave warm support and encouragement and put forward numerous useful suggestions. Professor Zhaoqian Xie and Dr. Guohai Chen reviewed this textbook carefully and provided valuable comments and advice. We would like to give our sincere appreciation to them.

We will appreciate comments and suggestions from the readers to improve the current edition. All constructive criticisms are welcome with gratitude.

Authors
January 2023

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Notations

A	area
b	width
c	generalized displacement of the support
d	panel
E	elastic modulus
f	arch height
F	force or generalized force
F_P	concentrated load
F_H	horizontal thrust
F_x, F_y	components of force in horizontal (x) and vertical (y) directions
F_N	axial force
F_{Nx}, F_{Ny}	components of axial force in horizontal (x) and vertical (y) directions
F_S	shear
F_S^L, F_S^R	shears at the left and right sides of section
F_S^F	fixed end shear
F_R	generalized reaction or reaction resultant
$\overline{\mathbf{F}}^e$	force vector of elemental member end in local coordinate system
$\mathbf{F}^e$	force vector of elemental member end in global coordinate system
$\overline{\mathbf{F}}_P^e$	fixed-end force vector of the element in local coordinate system
G	shear modulus
h	height
i	bending linear stiffness
I	sectional moment of inertia
$\mathbf{I}$	identity matrix
k	stiffness coefficient or nonuniform coefficient of shear stress distribution
$\overline{\mathbf{k}}^e$	elemental stiffness matrix in local coordinate system
$\mathbf{k}^e$	elemental stiffness matrix in global coordinate system

$\mathbf{K}$	structural stiffness matrix
l	length or span
m	mass or distributed bending moment
M	moment, couple moment or bending moment
M^F	fixed-end moment
n	degree of static indeterminacy
p	intensity of uniformly distributed load in horizontal direction
$\mathbf{P}^e$	nodal load vector of element
$\mathbf{P}$	nodal load vector of structure
q	intensity of uniformly distributed load in vertical direction
R	radius
r	radius or influence coefficient of reaction
S	static moment
t	temperature
$\mathbf{T}$	coordinate transform matrix
U	strain energy
u	horizontal displacement
v	vertical displacement or deflection
w	vertical displacement
W	work, computational degree of freedom, or weight
W_e	external virtual work
W_i	internal virtual work
X	generalized unknown force or generalized redundant unknown force
y	displacement
Z	response for influence line
α	linear expansion coefficient
Δ	generalized unknown displacement
$\mathbf{A}$	displacement vector
$\mathbf{A}^e$	nodal displacement vector of elemental member end
δ	flexibility coefficient or displacement influence coefficient
ε	linear strain
μ	Poisson ratio
κ	curvature
ϕ	angular displacement or chord angle
γ_0	mean shear strain
θ	rotational angle of section
ξ	elemental locating vector
ρ	material density
Π	total potential energy

Chapter 1

Introduction

Humans had built various kinds of buildings and structures since ancient times, such as the Pyramids in Egypt, the Gard Aqueduct in France, the Great Wall, Dujiangyan Irrigation System, the Zhaozhou Bridge, and the Forbidden City in China. With the advancement of science and technology, people gradually possessed some knowledge of the rules of structural design and the strength, stiffness, and stability of structures. According to the experience and experiments, a specialized discipline of structural analysis or structural mechanics was formed, which has been widely applied in practical engineering fields.

As far as the basic principles and methods are concerned, structural mechanics was developed simultaneously with theoretical mechanics and mechanics of materials. Therefore, structural mechanics was merged with theoretical mechanics and mechanics of materials in the early stage of development. In the early 19th century, owing to the development of emerging industries, people began to design a variety of large-scale engineering structures that require more accurate analysis and calculation. As a result, the theories and methods for analysis of engineering structures began to be independent. By the middle of the 19th century, structural mechanics started to become an independent discipline. Since the 20th century, owing to the emergence of new materials, new structures, new theories, and methods, the discipline of structural mechanics has been deepening in connotation and expanding in scope, which has developed into a disciplinary group covering the statics, dynamics, stability theory, computational structural mechanics, intelligent material structures, structural control, and health monitoring. Its application scope has also been extended to the fields of civil engineering, hydraulic engineering, mechanical engineering, aerospace engineering, naval architecture, ocean engineering, and so on. In fact, structural analysis is an ancient and ever-green discipline of applied mechanics, which is developing constantly with the times, and is a bridge linking the mechanics theory with engineering practice.

1.1 Research Object and Tasks of Structural Analysis

1.1.1 Research Object

Engineering structure means the part of the building, construction, or other civil infrastructure that bears and transmits the loads, which plays the role of the skeleton and is simply called a **structure**. For example, the beam-column system in buildings, the dams and gates in hydraulic structures, the highway and railway bridges, tunnels, culverts, and the skeletons of ships, automobiles, airplanes, and manned spacecraft subjected to the loadings, etc., are all typical examples of engineering structures (figure 1.1).

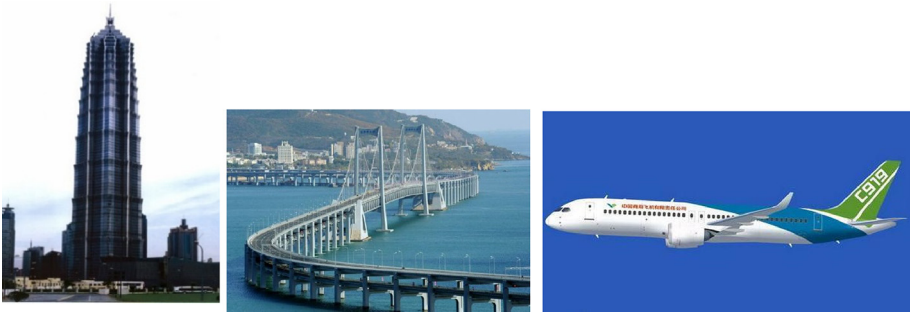


FIG. 1.1 – Building, bridge, and airplane.

The mechanical properties and load-bearing capacities of structures are closely related to their geometric characteristics. The structures can be divided into three categories according to geometric features.

- (1) **Member structure**—is formed by some interconnected members. The geometric feature of the member is that the cross-section dimension is much smaller than the length. A beam, arch, truss, and frame are typical forms of the member structure.
- (2) **Plate and shell structures**—are also known as thin-walled structures (figure 1.2a, b). Their geometric feature is that the thickness is much smaller than the length and width. The floor slab and shell roof in the building as well as the outer shell of the aircraft and the ship etc., all belong to the plate and shell structures.
- (3) **Massive structure**—also called a three-dimensional continuum structure. Its geometric feature is that the dimensions in the length, width, and height are similar. Gravity retaining walls (figure 1.2c) and gravity dams in hydraulic engineering are massive structures.

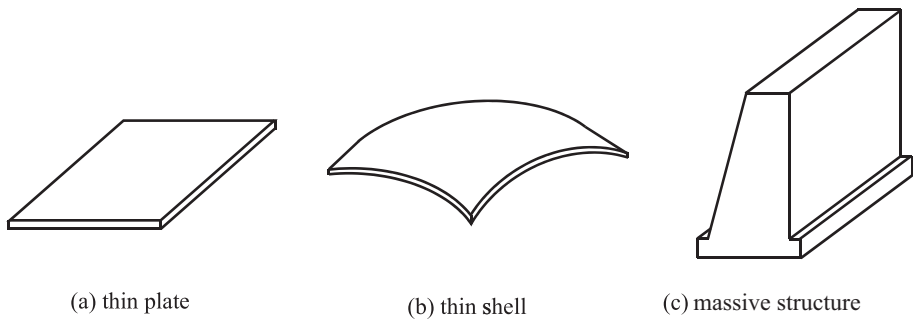


FIG. 1.2 – Thin-walled structure and massive structure.

In a narrow sense, the structure often refers to the member structure, and the usual structural analysis means the analysis of member structure.

Structural mechanics is closely related to theoretical mechanics, mechanics of materials, and elastoplastic mechanics. Theoretical mechanics mainly discusses the basic law of mechanical motion of particle (system) and rigid body (system), ignores the deformation effects of objects, and often regards the object as a particle or rigid body. The other three kinds of mechanics focus on the deformation effect of the object, and discuss the strength, stiffness, stability, and dynamic responses of structures and their members. Among them, the main research object is a single member in the mechanics of materials, and it is the system of member structure in structural analysis, while in elastoplastic mechanics the research objects are massive structure, plate, and shell structures.

1.1.2 Tasks

Structural analysis is a discipline that studies the rational forms of structures (such as beams, trusses, frames, arches, etc.) and the rules or laws of internal forces, deformations, dynamic responses, and stability of structures under external forces, and other external factors. The research purpose is to make the structure meet the requirements of safety, applicability, and economy. Specifically, the basic tasks of structural analysis include the following aspects.

- (1) Study the construction rules, mechanical properties, reasonable forms of the structures, and the reasonable choice of structural computational models.
- (2) Investigate the computational methods of internal forces and deformations of structures, and check the structural strength and stiffness.
- (3) Research the stability of structures and the structural responses and vibration control under dynamic loads.

The research methodology for problems of structural analysis includes theoretical analysis, numerical calculation, and experimental study. The course of structural analysis focuses on theoretical analysis and numerical calculation. In structural analysis, the actual structure is firstly simplified as a computational model, and then

the corresponding computation is performed. There are various kinds of computational approaches in structural analysis. These approaches take into account the following three types of fundamental equations.

- (1) Equilibrium equation or motion equation of the structural system.
- (2) Geometric continuous equation of deformation (or called compatibility equation).
- (3) Constitutive equation between stress and deformation (or called physical equation).

The basic solution method of structural analysis directly utilizes the above three types of fundamental equations for calculation and analysis, which can be called the **equilibrium-geometry-constitutive method**, for example, the force method and displacement method. The solution method is named an **energy method** such as the unit load method, or Rayleigh–Ritz method for displacement computation, if it is expressed with the form of virtual work or energy.

In the past, structural analysis problems were relatively simple and could be solved by analytical approaches and hand computation. With the advent and popularity of electronic computers, a large number of complex engineering analyses and design problems mainly relied on computer calculation. Computer calculation has enhanced the capability of solving problems and also has put forward a new requirement for structural analysis, that is, the computer method must adapt to the attribute of computer calculation. Therefore, some contents closely related to computer analysis of structures, such as energy principle, matrix displacement method, finite element method, discrete element method, meshless method, isogeometric analysis, structural analysis software, and structural optimum design, etc., have occupied an increasingly important position in structural analysis. In the field of structural analysis, energy principles, numerical methods, and program software have formed a new subdiscipline—computational structural mechanics, which is a branch of solving structural analysis and design problems by numerical methods with the aid of computers.

1.2 Computational Models of Structures

The engineering structures are usually very complicated, and it is neither possible nor necessary to perform mechanical analysis completely according to the actual situations. Thus, before implementing the mechanical analysis and structural calculation, the real structure is represented by an abstract and simplified model, which is called a **computational model of structure** or mechanical model by means of simplification and assumption, neglecting unimportant details and highlighting the essence and basic characteristics. Generally speaking, in structural analysis, the real structure is first simplified as a mechanical model, then the mathematical model, numerical model, and computer program are established, and finally the calculated results are applied to engineering analysis, design, and construction. This process is shown in figure 1.3, where the establishment and resolution of the associated

equations are two key links. Model verification and validation, which are closely related to structural analysis, are important research topics in the academic community. It is required that the theoretical methods of structural analysis should be verified by physical experiments and engineering practices.

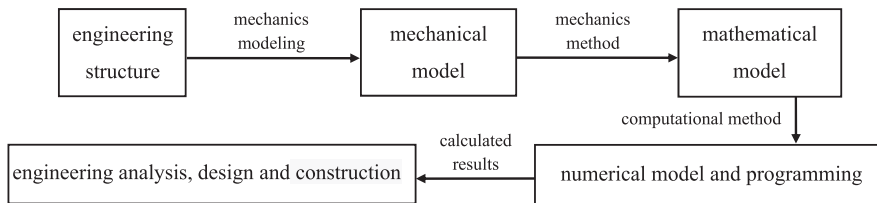


FIG. 1.3 – Process of computational modelling for structure.

The establishment of a computational model is the basis of mechanical computation and is extremely important. The established principles of structural computational model are presented as follows:

- (1) Consistent with the actual situation—the computational model should not only reflect the main force and deformation characteristics of the actual structure but also make the calculated results accurate and credible.
- (2) Easy to calculate—distinguishing the primary and secondary aspects, omitting the details, ignoring secondary factors, and making the analysis and computing procedure convenient and simple.

For constructing a computational model, it is necessary to simplify the real structure in some aspects. A brief description of the key points of simplification to establish the computational models of member structures is given as follows.

1.2.1 *Simplification of Structural Systems*

The member structure systems can be divided into plane member structures and space member structures. The general structures are actually space structures, where the parts are interconnected to form a spatial entirety in order to withstand the loads that may occur in all directions. However, in most cases, the actual structures can be decomposed into plane structures by ignoring some secondary spatial constraints, so that the computing procedure can be simplified. This textbook focuses on the computational issues of plane structures. Of course, there are also some structures that are not suitable to be simplified as plane structures because of their significant spatial characteristics and effects.

1.2.2 *Simplification of Members*

The cross-sectional dimensions (width and thickness) of members are usually much smaller than the length, and the stress on the section can be determined according to

its internal force (bending moment, axial force, and shear force). Therefore, in the computational model, the bars are represented by their axes, the connecting zones between the bars are represented by joints, the length of the bar is represented by the distance between joints, and the action points of the loads are also indicated at the axes.

1.2.3 Simplification of Joints

The interconnection zones of the bars are called **joints**. Usually, the joints are simplified into two ideal types:

- (1) **Pinned joint** The connected members cannot move with respect to each other at the joints but can rotate slightly against each other, *i.e.*, they can transmit forces rather than moments. This ideal situation is difficult to be realized in practical engineering. The joints of the timber truss are closer to the pinned joints (figure 1.4a, b).

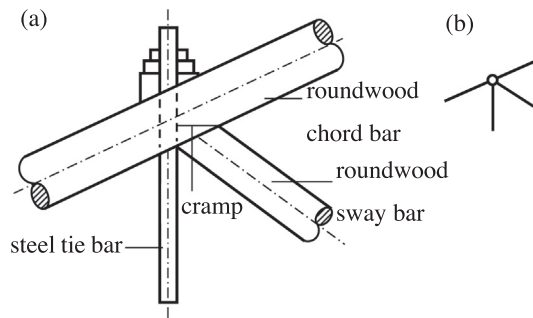


FIG. 1.4 – Pinned joint.

- (2) **Rigid joint** The connected members can neither move nor rotate with respect to each other at the joints. They can transmit both forces and moments. The cast-in-situ reinforced concrete joint usually falls into this category (figure 1.5a, b).

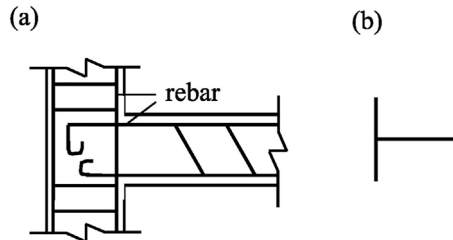


FIG. 1.5 – Rigid joint.

1.2.4 Simplification of the Supports

The connections between the structures and foundations are simplified as **supports**. According to mechanical characteristics, they are generally simplified into the following four types:

- (1) **Roller support** The supported part can rotate and translate horizontally, and cannot move vertically (figure 1.6a). The reaction provided by roller support is only the vertical reaction force F_y . The roller support is represented by a vertical support link in the computational model (figure 1.6b). Roller bearings and rocker bearings used in bridge structures are actual examples of roller supports.

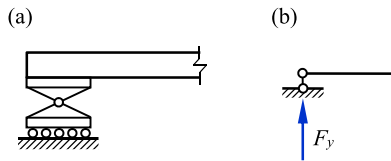


FIG. 1.6 – Roller support.

- (2) **Pin support** The supported part can rotate and cannot translate (figure 1.7a). The pin support can provide two reaction forces F_x and F_y , and is represented by two intersecting support links in the computational model (figure 1.7b).

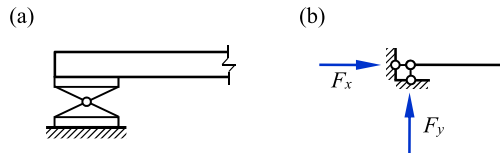


FIG. 1.7 – Pin support.

- (3) **Slider support** The supported portion cannot rotate, but can slide in parallel along one direction (figure 1.8a). The slider support can provide a reaction moment M and a reaction force F_y and is denoted by two parallel support links in the computational model (figure 1.8b).

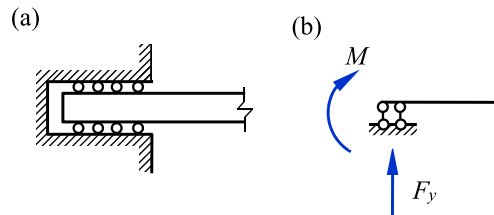


FIG. 1.8 – Slider support.

- (4) **Fixed support** The supported part is completely fixed (figure 1.9a). The fixed support can provide three reactions F_x , F_y , M , and is represented in the computational model as figure 1.9b.

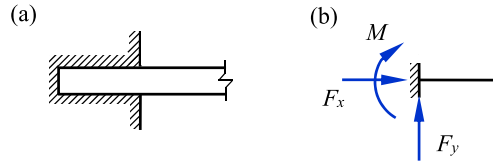


FIG. 1.9 – Fixed support.

1.2.5 Simplification of Material Properties

The materials used in the engineering structures are usually steel, concrete, brick, stone, timber, and so on. In structural computation, the materials for composing various members are generally assumed to be continuous, uniform, isotropic, elastic, or elastoplastic for simplification.

The above assumptions for metal materials with internal forces are in line with the actual situation in a certain range, whereas those for concrete, reinforced concrete, brick, stone, and other materials are approximations to a certain extent. As for timber, it is necessary to pay attention to the characteristic of anisotropy because its physical properties in the directions of grain and cross-grain are different.

1.2.6 Simplification of Loads

The loads applied on structures can be divided into body forces and surface forces. Body forces refer to the self-weight or inertia force of the structures, while surface forces are the acting forces transmitted to the structures by other objects through the contact surfaces, such as soil pressure, vehicle wheel pressure, and so on. The members are simplified as axes in member structures. Therefore, both the body forces and surface forces can be simplified as the loads acting on the axes of members. Moreover, the loads can also be simplified as concentrated loadings and distributed loadings in terms of their distributions.

To illustrate the simplification process of actual structures, an example of a single-story reinforced concrete industrial plant is exhibited in figure 1.10a. The beams and columns of the plant are prefabricated. The bottom ends of the columns are inserted into the cup rabbets of bases and then filled with fine aggregate concrete. The connections between the beam and column are realized by welding the embedded steel plates between the beam end and column top. Columns and beams form a framed bent in the transverse plane (figure 1.10b). The bent is connected by the roof panel and crane beam on the corbel.

The computational model shown in figure 1.10c can be applied to the mechanical analysis of the plant structure.

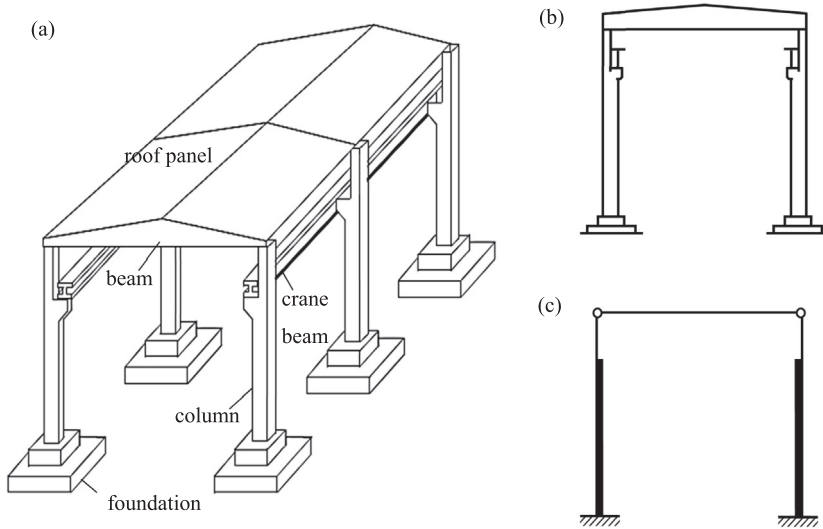


FIG. 1.10 – Simplification process of a single-story reinforcement concrete industrial plant.

First of all, although the plant structure is a space structure in which some framed bents are connected by roof panels and crane beams, the bents are regularly aligned with a certain interval in the longitudinal direction. Generally, the loads applied on the plant, such as dead load, snow load, and wind load, are evenly distributed along the longitudinal direction. Usually, these loads can be assigned to each of the bents, and each bent is regarded as an independent system. Hence, the actual spatial structure is simplified as a plane structure (figure 1.10b).

Then, the beams and columns are represented by their geometric axes. Since the sectional dimensions of beams and columns are much smaller than the length, the axes can be approximated as straight lines. Due to the difference in the sectional dimensions, the heavy line is used to indicate the thicker portion of the column.

The connections between beams and columns only depend on the welding of the embedded steel plates. Although the beam end and column top cannot translate relatively, it is possible that a slight relative happens. Hence, the connections between beams and columns in bent can be taken as the pinned joints. Because the relative translation and rotation cannot occur basically between the bottom of the column and the foundation, the bottom of the column can be viewed as fixed support.

1.3 Classification of Member Structures and Loads

1.3.1 Classification of Member Structures

The classification of structures actually refers to the classification of structural computational models. According to the mechanical characteristics, the member structures usually are divided into the following five categories:

- (1) **Beam** The beam (figure 1.11a) is a flexural member whose axis is usually a straight line. The beam has a single-span or multi-span beam.
- (2) **Arch** The axis of the arch (figure 1.11b) is a curve, and its mechanical characteristic is that there is a horizontal support reaction force (namely, thrust force) under vertical loads. Accordingly, the section of the arch has larger axial compression, smaller bending moment and shear force.
- (3) **Truss** The truss (figure 1.11c) consists of straight bars, whose junctions are pinned joints. When the truss is subjected to joint loadings, the sections of its bars only yield axial force.
- (4) **Frame** The frame (figure 1.11d) is also composed of straight bars, and the connections are usually rigid joints. The members of the frame are flexural members, and their internal forces include the bending moment, shear force, and axial force.
- (5) **Composite structure** The composite structure (figure 1.11e) is a combination assembled by the truss and beam or frame, which contains composite joints.

The above-mentioned five types of structures are the most fundamental structural categories. In addition, there are the suspension structure and cable-stayed structure, where the main load-bearing members are the cables hanging on the towers and columns with only the tension.

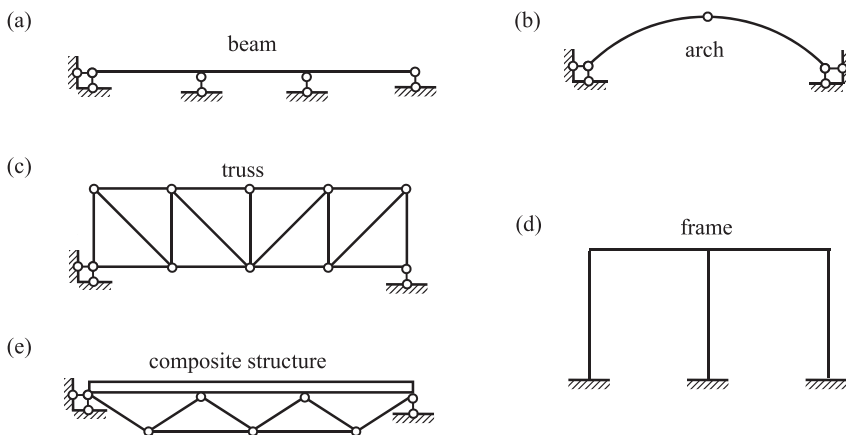


FIG. 1.11 – Various member structures.

According to the spatial position of the structural systems, the member structures are classified as plane structures and space structures. For plane structure, the axes of various members and action lines of external forces are all in the same plane, and figure 1.12 displays a plane truss structure. The space structure does not satisfy the above conditions. Figure 1.13 illustrates a space frame, where the axes of members are not in the same plane. In engineering design, most structures are usually viewed as plane structures for computation. In some cases, the spatial effect of structures must be considered.

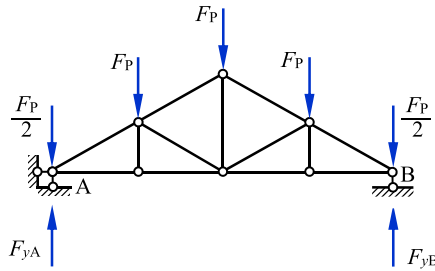


FIG. 1.12 – Plane truss.

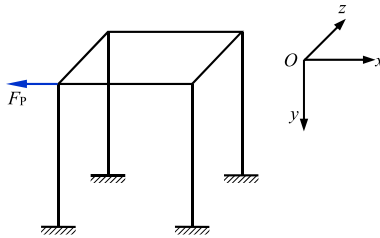


FIG. 1.13 – Space frame.

In addition to the above-mentioned classification, according to the computational characteristic, the structures are divided into statically determinate structures and statically indeterminate structures. The structures whose internal forces of members (including the support reactions) can be uniquely determined by the equilibrium condition are called **statically determinate structures**. If the internal forces of members and the support reaction forces cannot be uniquely determined by the equilibrium condition, and the deformation condition must be also considered, then the structures are called **statically indeterminate structures**.

1.3.2 Classification of Loads

Load is the external force actively applied to a structure, for example, the self-weight of the structure, the crane load in the industrial plant, the water pressure, and the

soil pressure acting on the structure. Besides the external force, other factors can cause internal force or deformation of the structure, such as temperature change, foundation settlement, material shrinkage, and so on. In a broad sense, these factors can also be called loads.

Before the calculation, the loading on the structure must be determined. The determination of load is an extremely important task in structural design. If the load is overestimated, the designed structure will be too bulky and wasteful. On the contrary, if the load is underestimated, the designed structure will not be safe enough. In terms of the action time, action position, and loading property, the loads can be classified into the following types.

(1) Classification according to the action time of the load

The loads can be divided into dead load and live load. Dead load is a constant load acting on the structure for a long time, such as the structural self-weight or soil pressure. Live load is a varying load temporarily acting on the structure, such as the moving train, crane load, crowd, snow load, and wind load.

(2) Classification according to the action position of the load

It is considered that the position of the dead load or most of the live load (such as snow load and wind load) acting on the structure is fixed, and such a load is named a **fixed load**. For a certain live load, *e.g.*, the load acting on a crane beam, or the vehicle load on a highway bridge, its position is mobile on the structure, which is called **moving load**.

(3) Classification according to the loading property

The loads are also divided into static load and dynamic load. The magnitude, direction, and position of **static load** do not vary or change very slowly with time, which does not cause significant acceleration of structure. Thus, the influence of inertial force can be neglected. The **dynamic load** is such a load that changes rapidly with time or suddenly acts or disappears in a short period, causing significant acceleration of the structure, and hence the effect of inertial force cannot be ignored. The self-weight and other dead loads of the structure are static loads. The load generated by the machine and the pressure of shock wave are examples of dynamic loads. Wind load and earthquake action are usually considered as dynamic loads, but sometimes they are simplified to static loads in structural design.

1.4 A Brief History of the Development of Structural Analysis

In the 17th century, several giants in the history of science laid the cornerstone of modern science—classical mechanics. In 1609 and 1619, J. Kepler (1571 ~ 1630) successively published *New Astronomy* and *Cosmic Harmony Theory* and proposed Kepler's three laws of planetary motion. In 1632, G. Galileo (1564 ~ 1642) published *Dialogues on Two World Systems of Ptolemy and Copernicus*, which supported

heliocentrism and firstly clarified the principle of relativity of the movement. In 1638, he published *Dialogues on Two New Sciences*, which discussed the problems of fracture resistance of materials, the resistance of medium to motion, the principle of inertia, the motion of free fall, the motion of an object on a slope, motion of a projectile, etc., and gave the definitions of uniform motion and uniform accelerated motion. In 1687, I. Newton (1643 ~ 1727) summed up the achievements of predecessors and published *Mathematical Principles of Nature Philosophy*, which expounded and established Newton's laws of motion and the law of universal gravitation, as well as constructed the basic framework of classical mechanics. Structural analysis was gradually developed under the framework of classical mechanics.

1.4.1 Energy Principles and Energy Methods

Energy principles are the theoretical foundation of structural analysis, from which several universal approaches of displacement calculation and internal force analysis, namely the energy methods, were derived. The principle of virtual displacements was proposed by J. Bernoulli (1667 ~ 1748) in 1717. British scientist J. C. Maxwell (1831 ~ 1879) established the theorem of reciprocal displacements in 1864 for a simple case with only two forces and proposed a unit load method for computing displacement. Subsequently, Italian scholar E. Betti (1823 ~ 1892) fulfilled a universal proof of Maxwell's theorem of reciprocal displacements in 1872 and generalized it to the theorem of reciprocal works. A. Castigliano (1847 ~ 1884) proposed Castigliano's first and second theorems and the principle of minimum work (*i.e.*, the principle of strain energy minimization or the principle of minimum potential energy) in 1879. German scholar F. Engesser (1848 ~ 1931) advanced the concept of complementary energy in 1884 and devised the method of complementary energy in 1889.

The principle of virtual work for deformable bodies means that external virtual work equals internal virtual work, which contains the principle of virtual forces and the principle of virtual displacements. From the principle of virtual forces for a deformable body, a general method for calculating the displacement or rotational angle of the specified point of beam, frame or truss can be derived, that is, the unit load method. As mentioned above, in 1864 Maxwell proposed the unit load method. In 1874, German mechanician O. Mohr (1835 ~ 1918) also independently established this method. Therefore, the unit load method is also called the Maxwell–Mohr method. Both of them had once applied the unit load method for computing the displacements of statically determinate trusses. The unit load method utilizes three basic conditions, namely, the equilibrium condition of the force, compatibility condition of deformation, and physical condition (or constitutive relation). Since the physical condition is not limited to describing linear elasticity, the unit load method is also suitable for solving the displacements of inelastic structures. Moreover, this method is applicable for both statically determinate and statically indeterminate structures. For any balanced rigid body system with ideal constraints, its internal virtual work is zero. Therefore, the virtual work done by the active forces (external

forces) of the rigid body system is zero, which is the virtual work principle of a rigid body. From the principle of virtual displacements for rigid body systems, an efficient and convenient method for calculating the specific constraint force of statically indeterminate structures was developed, that is, the unit displacement method.

In 1879, Italian railway engineer A. Castigliano published a monograph on the *Equilibrium Theory of Elastic System*, and based on the principle of virtual work, he advanced Castigliano's first and second theorems and the principle of minimum work. Castigliano's first theorem leads to the approach for determining the constraint force of a section after the displacements of the structure occur. This approach is appropriate to solve the forces of statically indeterminate structures. Its application is independent of the superposition principle and can be extended to nonlinear elastic systems, and it is equivalent to the force equilibrium condition of the structure. Castigliano's second theorem leads to the approach to solving the displacement of a point in trusses, beams, or frames. It can be extended to the nonlinear elastic structures and is equivalent to the compatibility condition of deformation. According to Castigliano's second theorem, the displacement at a point is equal to the first-order partial derivative of strain energy with respect to the force acting at this point along the direction of displacement to be solved. Castigliano's second theorem is used to determine the displacement of structures at a certain position on the premise that the load needs to apply on this position corresponding to the displacement to be solved. If there is no load corresponding to the displacement to be computed, the method of additional force can be employed. In 1878, F. Crotti proposed the theorem of calculating the displacement of elastic body. In 1889, F. Engesser also independently devised this theorem, which is called the Crotti-Engesser theorem. This theorem is derived based on the principle of virtual forces and the concept of strain complementary energy, and it is applicable to the displacement calculation of nonlinear elastic structures. Castigliano's second theorem is a special case of the Crotti-Engesser theorem. Hence, the principle of minimum potential energy and the principle of minimum complementary energy were established successively.

In 1886, German mechanician H. Muller-Breslau (1851~1925) developed a method for quickly determining the shapes of influence lines of internal forces of statically determinate and statically indeterminate beams based on the principle of virtual displacements. Namely, the influence line of a certain quantity (force) of interest is the same as the deflection shape of the released beam under this force. This method is called the Muller-Breslau principle or Muller-Breslau criterion, which is called the kinematic method in textbooks. In 2015, Dixiong Yang *et al.* established a kinematic method for analytically, exactly, and conveniently calculating the influence line equations of internal forces for statically indeterminate beam structures by utilizing the principle of virtual displacements for the deformable body, which overcomes the drawback of cumbersome computation of the past conventional approach for constructing influence lines of statically indeterminate forces.

In the middle of the 20th century, breakthrough progress had been made in the aspects of energy principles and energy methods. The researchers successively proposed the generalized variational principles with the two and three classes of

variables, which are different from the univariate variational principle (*i.e.*, the principle of potential energy and the principle of complementary energy), and provide a new approach for structural analysis. In 1950, E. H. Reissner advanced the generalized variational principle with two classes of variables in elasticity theory, showing the prospect of establishing equations that simultaneously satisfy different mechanical properties in the energy method. In the same year, Lingxi Qian (1916~2009) published a paper on the *Principle of complementary energy* in *Science China*, which pioneered the study of variational principles in the Chinese community of mechanics. Subsequently, some researches with international influence on variational principles appeared in China, a few of which were inspired by Lingxi Qian's paper. For instance, the generalized variational principle with three classes of variables was proposed by Haichang Hu (1928~2011) in 1954, and similar work was also published by K. Washizu in the United States in 1955, which was called Hu–Washizu variational principle.

1.4.2 Force Method and Displacement Method

Before the development of the force method and displacement method, one can only calculate the internal forces and displacements of simple structures. In the several decades since 1847, graphic methods (Maxwell diagram, etc.) and analytical methods (method of joints and method of sections) were applied to study the force analysis of statically determinate truss structures, and the foundation of truss theory was established. The force method was first proposed by Maxwell in 1864, then amended by Mohr in 1874, and fundamentally developed by Muller-Breslau in 1886. The basis of the force method is the compatibility condition of deformation, thus it is a compatibility method or flexibility method in essence. When the internal forces of statically indeterminate structures are solved by the force method, the redundant constraint forces are taken as the primary unknowns. Firstly, the geometrically unchangeable primary system is selected, and the canonical equations of the force method are established by the condition of deformation compatibility. Then, the influence coefficients of displacement in the equations of force method are obtained by the unit load method. Finally, the equations are solved to get the redundant forces. When the redundant forces are determined, the other support reactions and internal forces of statically indeterminate structures can be calculated by the static equilibrium condition.

The basic principle of the force method is simple and easy to understand. Nevertheless, for the complex statically indeterminate structures, the selection of the primary system requires more experience and intervention of users. Moreover, for the multistory multi-span frame structures, the number of redundant unknown forces is greater, and the difficulty of solving equations increases. The displacement method can overcome these difficulties. The development of the displacement method is divided into two stages of the slope-deflection method and the matrix displacement method. In 1826, the French mechanician C. L. M. H. Navier (1785~1836) advanced the displacement method in elasticity mechanics and applied it to solve the internal forces of statically indeterminate trusses.

The secondary bending moment method as the predecessor of the slope-deflection method was first proposed by the German H. Manderla in 1880 for computing the secondary bending stress of the truss, and it was modified by Mohr in 1892 and gradually became known. The secondary bending moment method assumes that the moment does not affect the joint displacements of truss structures, which is only effective for small bending moment effects. In the 20th century, with the gradual increase of bridges and high-rise buildings built with reinforced concrete material, the bending moment effect caused by the rigid joints became significant, and the joint displacements yielded by the axial forces were small. In 1914, Danish engineer A. Bendixen applied the slope-deflection method for calculating the internal forces of the frames with sidesway. In 1915, American scholars W. M. Wilson and G. A. Maney modified the secondary bending moment method and independently used it to solve the internal forces of frames, and called it the slope-deflection method. In 1926, Danish scholar A. Ostenfeld pointed out the duality between the force method and the displacement method. In fact, the canonical equations with respect to unknown displacements in the structures are complementary to the canonical equations for redundant forces. When using the displacement method to solve the internal forces of the structure, the independent rotational angles (slopes) and linear displacements (deflections) are taken as the primary unknowns, and the whole structure is divided into a number of single members. Based on the force-displacement relation (the slope-deflection equation) of the members, the canonical equations of the displacement method are constructed by the equilibrium condition of joint moments and section projection, and then the displacements of joints are obtained by solving the equations. Finally, the displacements of joints are back substituted into the slope-deflection equations of members to obtain the internal forces of member ends. The displacement method does not need to care whether the primary system is geometrically unchangeable or not (note that the primary system of the force method cannot be a geometrically changeable system), and it is suitable for the mechanical analysis of statically determinate and statically indeterminate structures.

It is seen that both the force method and the displacement method utilize the static equilibrium condition, the deformation compatibility condition, and the force-displacement relation (which can be analogous to the stress-strain relation of differential elements) of the structure. The difference lies in that for the force method, the force-displacement relation is constructed firstly by the unit load method at the level of the primary system (the flexibility influence coefficients representing the displacements of the primary system under the unit load), and then the deformation compatibility condition is satisfied and the flexibility equations of the force method are established at the structural level. Finally, the internal forces of member ends are solved *via* the equilibrium condition. While for the displacement method, firstly the deformation compatibility condition is considered at the level of the member element to express the force-displacement relation (the slope-deflection equation or the stiffness influence coefficients representing the internal forces of the primary structure under the unit displacement). Subsequently, the static equilibrium condition is met at the structural level to form the stiffness equations of the displacement method, and the obtained joint displacements are used to solve the

internal forces of member ends according to the deformation compatibility condition and force–displacement relation.

It is interesting to note that, there exist close connections between the force method, the displacement method, and the energy principles. The equation of the force method can be derived from the principle of complementary energy or Castigliano's second theorem, and the equation of the displacement method can be deduced from the principle of potential energy. Moreover, the development of the finite element method has a direct and profound connection with the variational principle. In general, the displacement method is suitable for solving statically indeterminate and determinate structures, while the force method is generally applicable to indeterminate structures. The displacement method can directly solve structural displacements and internal forces, while the force method cannot straightforwardly determine displacements. Additionally, the displacement method is easier to establish the stiffness equations and is convenient for the computer to solve automatically. Therefore, the displacement method is the key method of structural analysis. It is worth pointing out that the force method introduced in the textbook takes redundant constraint forces as unknowns, while some efforts are also being made to develop the integrated force method, which is parallel to the displacement method and takes the whole forces as unknowns.

Before the advent of the computer and matrix displacement method, it was very troublesome for engineers to solve the higher-order algebraic equations obtained by the slope-deflection method. In 1922, Calisev proposed a successive approximation method by taking the rotational angles of the frames without sideways as unknowns, which can avoid directly solving the higher-order equations. In 1930, American scholar H. Cross developed a successive displacement method named the moment-distribution method, to approximately solve the statically indeterminate continuous beams. In fact, the ideas of the successive approximation method and the moment-distribution method are almost the same. The calculation process of the moment-distribution method is to constrain and relax the joints one by one, distribute and balance the bending moments at the joints in turn, and repeat the iteration until the unbalanced bending moments or the constraint moments of the joints are close to zero. The moment-distribution method applies successive iterative strategies to solve the simultaneous algebraic equations of the displacement method, which overcomes the difficulty of directly solving higher-order equations. It was a successful application of the iterative method for solving linear algebraic equations in structural mechanics, which attracted great attention in the engineering community and was the most significant progress in structural analysis in the 1930s. However, after the invention of the computer, the moment-distribution method and other successive analysis methods, such as the moment-iteration method, the no-shear distribution method, and the Southwell relaxation method, gradually retreated from the center of the historical stage and gave way to the matrix displacement method which emerged in the 1950s.

1.4.3 *Matrix Displacement Method and Finite Element Method*

Firstly, the development of the finite element method is introduced, and then a specific description is given. The idea of the finite element method had been explicitly proposed by mathematician R. Courant (1888~1972) in 1943. In the 1950s, J. H. Argyris (1913~2004) and R. W. Clough (1920~2016) *et al.* developed the finite element method from aircraft structural analysis. In 1960, Clough coined the term “finite element”. Many researchers such as Chinese mathematician Kang Feng (1920~1993), British mechanician O. C. Zienkiewicz (1921~2009), and American scholar Xuehuang Bian (1919~2009) had made important contributions to the development of the finite element method.

In 1943, Courant first applied the piecewise interpolation function defined on the triangle region and solved the Saint–Venant torsion problem in conjunction with the principle of minimum potential energy. Zienkiewicz wrote in a review article: “It is a pity that, not being an engineer, he did not link the idea with networks of discrete elements and that, for this reason, I believe, his work had to lie in obscurity for so many years.” Later, the great significance of Courant’s work was recognized, and the year 1943 was regarded as the birth year of the finite element method.

After the 1940s, aeronautical engineers needed to perform detailed stress analysis of continua such as plates, frames, beams, and trusses in metal aircraft structures, and the finite element method originated from the structural analysis of these continua. Greek mechanician Argyris published a series of papers in the journal *Aircraft Engineering* from October 1954 to May 1955. They were reprinted as a monograph entitled *Energy Theorems and Structural Analysis* in 1960. This monograph contains the following two main contributions. Firstly, Argyris generalized and unified the basic energy principles of elastic structure analysis. Secondly, he developed a practical method for complex structural analysis of aeronautical engineering, namely, the matrix analysis method, including the matrix displacement method and matrix force method. The matrix displacement method establishes the elemental stiffness matrices by means of the slope-deflection equation, and assembles them into the global stiffness matrix to form the global stiffness equation of the displacement method, then solves the equation to obtain the displacements of nodes, and further achieves the internal forces of members through the back substitution into the elemental stiffness equation. The matrix displacement method is the prototype of the finite element method and can be derived from the principle of minimum potential energy. It can also be called the finite element method of member structures. The attributes of the matrix displacement method lie in that, the establishment of the primary system is simple and uniform, the formation of elemental and global stiffness matrices is modular, and the solution of equations is easy to program. Accordingly, the matrix displacement method fairly conforms to the requirement of computer automatic calculation and is widely applied owing to its generality. The matrix force method was no longer used because it inherited the disadvantages of the force method requiring the selection of a primary system and possessed the limitation that it is difficult to calculate automatically. In 1965, the *Theory of Structures* (2nd Edition) co-authored by S. P. Timoshenko (1878~1972)

and Young introduced the matrix force method and the matrix displacement method.

In the summer of 1952, Clough came to Boeing company to work on the structural analysis of the aircraft wing under the guidance of M. J. Turner and attempted to calculate the influence coefficients of the box beam with a small aspect ratio, but the displacement value obtained by Levy method differed from the measured value up to 13% ~ 65%. In the summer of the following year, Clough returned to Boeing, and Turner suggested a better way to model the skin panel. Clough proposed the Ritz analysis method in combination with the simple strain field. In 1956, Turner and Clough *et al.* published a famous paper on the analysis of two-dimensional elastic structures, in which they utilized three types of elements, namely triangle, quadrilateral and rectangular elements, to discretize the continuum. Actually, they advanced the finite element method which adopted the idea of the direct stiffness method (matrix displacement method) for the analysis of member structures, and the finite element used was the displacement element. The elemental stiffness matrices and the structural global stiffness equation were established in that paper. This work is considered to be the beginning of the finite element method in the engineering community. In 1960, Clough first named the finite element and opened up a broad way to transform the continuum mechanics problems into discrete mechanical models. E. L. Wilson, as Clough's early Ph.D. student, made an important contribution to computer simulation based on the general program of structural analysis and developed the currently popular SAP software system.

In the early 1960s, Kang Feng independently and separately developed the mathematical theory of the finite element method to address the stress analysis problem of the dam. In 1964, he established the finite element method for numerically solving the elliptic partial differential equations, which formed a standard algorithm, and compiled a general computer program for engineering structure analysis. In 1965, an article on *Difference schemes based on variational principles* was published. The properties of convergence and stability of the method were proved under broad conditions. Kang Feng is a computational mathematician with an international reputation. He also developed the natural boundary element method and proposed the symplectic geometric algorithm for solving the Hamilton system, which opened a new way for the long-term structure-preserving computation of dynamic systems.

From 1963 to 1964, the study of J. F. Besseling, R. J. Melosh, and R. E. Jones indicated that the foundation of the finite element method (still called the direct stiffness method in the paper) is the variational principle. They proved that the finite element method is another formalism of the Ritz analysis method based on the variational principle, and confirmed that it is a universal method for dealing with continuum problems. Their research built the bridge of communication between the mathematics and engineering circles for a better understanding of the finite element method and pushed people to realize that it is a numerical method with both a rigorous theoretical basis and a common application value, and thus promoted the development of the finite element method.

In 1967, O. C. Zienkiewicz and Y. K. Cheung published the first book about the finite element method in the world, *The Finite Element Method in Structural Mechanics*, and later adapted and published the three-volume book *The Finite*

Element Method with R. L. Taylor. The seventh edition of this book was published in 2013, which made a great contribution to the development and dissemination of the finite element method. Zienkiewicz was engaged in the research of finite element method for a long time, and his outstanding contributions include that: he proposed the patch test approach and the isoparametric element, developed the error estimation method, as well as promoting the application of the finite element method in fluid mechanics and civil engineering.

To sum up, the finite element method was a novel numerical method for solving problems in continuum mechanics, physics, and various kinds of engineering sciences developed in the 1950s and 1960s. Specifically, the infinite degrees of freedom of the continuum are discretized into finite degrees of freedom through the subdivision of finite elements, so that the problem is reduced to solving algebraic equations based on variational principles or other methods. The finite element method not only has the advantages that its theory is complete and reliable, its formulation is simple and standard, and its precision and convergence are guaranteed, but also can construct appropriate elements according to the nature of the problem. Hence, the finite element method has a wider application range than other numerical approaches. The marriage of the finite element method and the computer has produced tremendous power, and it has become an indispensable tool for scientific research and technology development in many fields. Its application range is quickly extended from simple bar, beam, and plate structures to complex spatial composite structures. The finite element method makes the static analysis of some large-scale complex structures impossible in the past to become routine calculation, and numerous dynamic problems and nonlinear problems also have corresponding solution schemes. For the mechanicians, with the help of the finite element method, credible numerical results of many problems difficult to obtain analytic solutions can be achieved. For engineering and technology professionals, the design of many complex engineering systems can not rely on or less depend upon expensive experiments. At present, the finite element method has become a crucial method to solve mathematical physics equations and is popularly used in computational mathematics, computational physics, computational mechanics, electromagnetics, heat transfer, meteorology, geophysics, engineering science, and other fields. Indeed, the finite element method has truly revolutionized current engineering and scientific environments.

From the magnificent development history of structural analysis, we can not only understand the evolution process of mechanical theories and methods driven by the demand of engineering application, but also can appreciate the beauty of mechanics and the power of scientific innovation, and thus inspire the passion of scientific discovery and creation.

1.5 A Brief Introduction to Important Figures in Structural Analysis

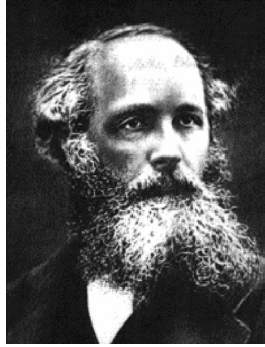
In the long history of the development of structural analysis, many important figures had emerged, who had made outstanding scientific contributions. The following

introduction highlights six great figures: C. L. M. H. Navier, J. C. Maxwell, O. Mohr, H. Müller-Breslau, Stephen P. Timoshenko, J. H. Argyris.



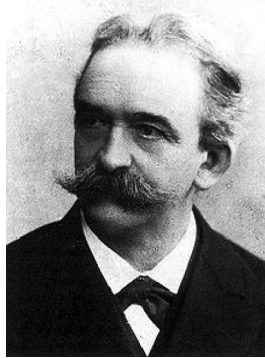
C. L. M. H. Navier (1785~1836), was a French mechanician and engineer. In 1802, he entered the Paris Polytechnic School for study. Then, he entered the Bridges and Roads School in 1804 and graduated in 1806. He taught applied mechanics at the Bridges and Roads School since 1818 and was a professor from 1831. In 1824, he was elected as a member of the French Academy of Sciences.

Navier's main contribution lies in that he established the basic equations for fluid mechanics and elasticity mechanics, respectively. In 1821, he generalized Euler's equation of fluid motion, took into account the forces between the molecules, and thus constructed the basic equations of fluid equilibrium and motion, which contain only one viscous constant. In 1845, Stokes improved his motion equation of fluid dynamics from the continuum model and obtained the form of rectangular coordinate components of fluid motion equation (later called the Navier-Stokes equation) with two viscous constants. In 1821, Navier also derived the equilibrium and motion equations of elastic solids from the molecular model (published in 1827). These equations contain only one elastic constant. The basic equations of isotropic elastic mechanics with two elastic constants were derived by Cauchy in 1823. In 1826, he proposed the idea of the displacement method in elasticity mechanics and used it to solve the internal force of a statically indeterminate truss. In addition, Navier's other achievements in mechanics include the following: he first (in 1820) utilized double trigonometric series to solve the fourth-order partial differential equation of simply supported rectangular plate and introduced mechanical work in engineering to measure the efficiency of the machine. In civil engineering, he changed the tradition of designing suspension bridges by only relying on experience, and adopted theoretical calculations in structural design.



James Clerk Maxwell (1831 ~ 1879), was a British physicist. In 1854, he graduated from Trinity College, Cambridge University, and earned a degree in mathematics. In 1856, he became a professor of natural philosophy at Marischal College in Aberdeen, Scotland. In 1860, he was a professor of natural philosophy and astronomy at London King's College. In 1861, he was elected as a member of the Royal Society of London. In the spring of 1865, he resigned from his faculty and returned to his hometown to systematically summarize the research results in electromagnetics and complete the classical great work *Treatise on Electricity and Magnetism* about electromagnetic field theory, which was published in 1873. In 1871, he was appointed as the newly established Cavendish experimental physics professor at Cambridge University and was responsible for the preparation to establish the famous Cavendish Laboratory, which was built in 1874 and he served as the first director until his death in 1879.

The most significant achievement of Maxwell is that he laid the foundation of classical electromagnetic theory. He also made important contributions to other fields of physics, such as thermodynamics, statistical physics, molecular physics, and optics. Maxwell's contribution to mechanics mainly includes the following aspects: in 1853, he generalized the method of measuring stress by polarized light; in 1864, he established the theorem of reciprocal displacements, and proposed the graphical method for solving the internal forces of trusses in structural analysis; he pointed out that the truss shape figure and the internal force figure are reciprocal, advanced the unit load method (also called Maxwell–Mohr method) for calculating the displacements of statically indeterminate trusses, and also developed the force method for analysis of indeterminate structures at the same time. In 1868, he proposed a model for characterizing viscoelastic materials (later referred to as the Maxwell model) and conceived the concept of relaxation time. In the same year, in his article *On governors*, the problem of motion stability of automatic governors of steam engines and clock mechanisms was investigated. In 1870, Maxwell extended the stress function in elasticity theory proposed by G. R. Airy from a 2-dimension to a 3-dimension case and pointed out that it should satisfy the biharmonic equation. In 1873, he gave the stress field caused by attractive and repulsive force in the electrical charged system.



Christian Otto Mohr (1835 ~ 1918), was a German mechanician and civil engineer. In 1851, he studied at the Hannover Institute of Technology and worked in the railway sector in Hannover and Oldenburg since 1855. He designed some famous bridges and promoted the early application of steel truss structures. In 1867, he became a professor of mechanics at the Stuttgart Institute of Technology, and in 1873 he taught at the Dresden Institute of Technology. His lectures were straightforward and unpretentious, and he was popular with the students.

Mohr had made great achievements in the fields of structural analysis and mechanics of material. In 1874, he independently developed the Maxwell–Mohr method of structural analysis, which can solve the nodal displacements of the trusses, also known as the virtual force method or the unit load method. The Maxwell–Mohr method can also determine the internal forces of the bars of statically indeterminate trusses, and it is a force method in this situation. Mohr was keen on developing the graphical method of mechanical analysis. In 1882, based on the Carl Culmann stress circle, he proposed the Mohr circle characterizing the 2-dimensional and 3-dimensional stress states, and thus established the shear stress-based strength theory. He also constructed the Williot–Mohr diagram for solving the truss displacements. In addition, Mohr and Emil Winkler independently suggested the concept of the influence line.



Heinrich Müller-Breslau (1851 ~ 1925), German mechanician and engineer. He was born in Breslau, Germany in 1851, and his surname was changed by himself to

avoid resemblance to others. After graduating from the Realschule in 1869, he became an engineer of the Prussian army and participated in the Franco-Prussian War from 1870 ~ 1871. Later, he studied civil engineering and mathematics at the Berlin School of Architecture and the University of Berlin. From 1875, he worked as an independent consulting engineer mainly working on the design of iron bridges. In 1885, he became a professor of civil engineering at the Hannover Institute of Technology. A series of published papers and books as well as some famous engineering designs quickly established his good reputation. In 1888, he succeeded Emil Winkler as the structural engineering director of the Berlin Institute of Technology and later became the dean.

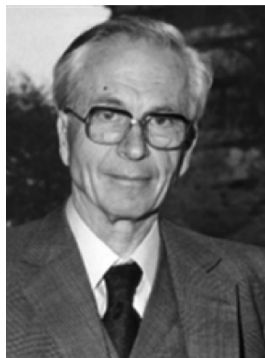
Müller-Breslau is considered to be the founder of modern structural engineering in Germany and made important contributions to the theory of beam and frame analysis in structural analysis. His numerous works refined and elaborated the existing theoretical methods and expressed them systematically and uniformly, and had a profound impact. He developed fundamentally the force method for analysis of statically indeterminate structures and proposed the kinematic method for drawing the influence lines of internal forces of statically determinate structures under moving load, and resolved many problems in structural theory, including the analysis of cantilever beam, arch structure, and lattice structure, soil pressure problems of retaining walls, and stability issue of straight members. His three volumes of handbook *Graphical Statics of Structural Theory* was the authoritative work about graphic mechanics of graphic mechanics which achieved international recognition. The major projects designed by Müller-Breslau included the Volga River bridge in Kazan, Russia, and the new cathedral in Berlin. He also participated in the manufacturing of airships, designed large-scale aircraft hangars, and introduced cantilevered wings to aircraft design.



Stephen P. Timoshenko (1878 ~ 1972), Ukrainian American engineering mechanician, and founder of modern engineering mechanics. He was born in Ukraine in 1878 and graduated from the Institute of Engineers of Ways of Communication in St. Petersburg in 1901. He worked in the Institute of Engineers of Ways of Communication, St. Petersburg Polytechnic Institute, Kiev Institute of Technology, Zagreb Institute of Technology, Philadelphia Vibration Company in the USA,

Westinghouse Electric Company, University of Michigan, and Stanford University. He was engaged in the study of structural analysis, mechanics of materials, and elasticity theory, made great groundbreaking contributions in many aspects, and was known as the “father of modern engineering mechanics”. In order to remember his great achievements in engineering science and education, the American Society of Mechanical Engineers established the Timoshenko Medal in 1957, and he was the first recipient. He received numerous honors and awards, and was elected as a member of the seven National Academy of Sciences: Ukrainian Academy of Sciences (1918), Russian Academy of Sciences (1928), Polish Academy of Sciences (1935), French Academy of Sciences (1939), National Academy of Sciences of USA (1941) The Royal Society of London (1944), Italian Academy of Sciences (1948).

Some works of Timoshenko in structural analysis, mechanics of materials, and elasticity theory were written into standard textbooks and had become the fundamental knowledge in the field of engineering mechanics. In particular, he developed the energy method of structural stability analysis and established the theory of transverse vibration of deep beams, namely the theory of the Timoshenko beam. Also, he proposed the concept of shearing center of beam and the theory of warping and torsion of thin-walled members, and promoted the development of elasticity mechanics. He published 12 textbooks, including *Strength of Materials*, *Elements of Strength of Materials*, *Theory of Elasticity*, *Theory of Plates and Shells*, *Theory of Elastic Stability*, *Theory of Structures*, *Vibration Problems in Engineering*, *Advanced Dynamics*, and *History of Strength of Materials*. These classical books of mechanics have been translated into 35 languages and distributed around the world, and are still in use today. These books are characterized by conciseness and clarity, describing complex phenomena and problems in straightaway language, often starting with simple and specific problems and generalizing them into more complex situations. In fact, his teaching style is similar, which is deeply beloved and welcomed by many students.



John Hadji Argyris (1913~2004), Greek mechanician. He was born in Volos, Greece in 1913. After studying civil engineering at the National Technical University of Athens, he obtained his Engineering Diploma at the Technical University of Munich in 1936. He spent his early career in the design and research of complex

structures and continued his studies in aeronautical engineering at the Swiss Federal Institute of Technology (ETH Zurich) from 1941 to 1942. In 1943, he joined the Engineering Department of the Royal Aeronautical Society of London. Since 1949, he was a senior lecturer in aeronautical engineering at the Imperial College of the University of London. In 1955, he was promoted to professor and director. In 1959, he became the director of the Institute for Statics and Dynamics of Aerospace Structures at the University of Stuttgart, Germany. Together with Ray W. Clough and Olgierd C. Zienkiewicz, Argyris was the leading figure in developing the finite element method following Richard Courant. He proposed the matrix force method and matrix displacement method for structural analysis, made far-reaching and fundamental contributions to computational mechanics and its industrial applications, and promoted the research of chaos theory. He was the author of: *Energy Theorems of Structural Analysis*, *Introduction into the Finite Element Method*, *Dynamics of Structures*, *An Exploration of Chaos*.

He was elected as the Fellow of the Royal Society of London (1986), and a Fellow of the Royal Academy of Engineering (1990). In addition, he received many awards, including the Gauss–Newton Award from the International Association for Computational Mechanics (1990), the von Karman Award from the American Society of Civil Engineers (1975), the Timoshenko Medal from the American Society of Mechanical Engineers (1981), and the Royal Medal of the Royal Society (1985), the Prince Philip Gold Medal of Royal Academy of Engineering (1996), the Grand Cross of Merit of the Federal Republic of Germany (1989), the Einstein Award from the Einstein Foundation, USA (2000).

Chapter 2

Kinematic Analysis of Plane Member Systems

A member system is composed of multiple members who are connected in a certain way. If all the members, constraints, and external loads of the system are in the same plane, the system is called a plane member system. From the viewpoint of the kinematic analysis, when the member deformation is not considered, member systems can be divided into two categories: the geometrically unchangeable (or stable) system and the geometrically changeable (or unstable) system, depending on whether the geometrical positions and the shape of the system are changeable when subjected to arbitrary loads. If it is changeable, the system is named a **geometrically changeable system** (such an example is the hinged three-bar system shown in figure 2.1a), and a mechanism is a geometrically unstable system. Otherwise, the system is named a **geometrically unchangeable system** (such an example is the system shown in figure 2.1b). Generally, the geometrically changeable system cannot be used as a structure, and only the geometrically unchangeable system can be employed as a structure to support and transmit loads.

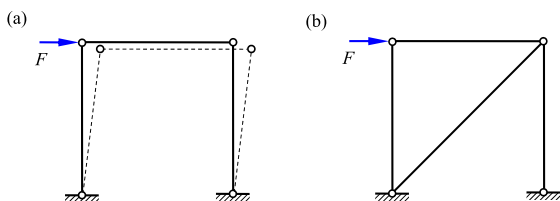


FIG. 2.1 – (a) Geometrically changeable system and (b) geometrically unchangeable system.

The **kinematic analysis** is also termed the geometric construction analysis, which focuses on analyzing the geometric stability of member systems according to their construction rules. The objectives of the kinematic analysis are: (1) to evaluate if a system is geometrically unchangeable, and to determine whether it can be used as a

structure; (2) to study the construction rules of a geometrically unchangeable system, so that the designed structure can support and transmit loads independently; (3) to provide the guidance for the force analysis of statically determinate and indeterminate structures.

In this chapter, basic concepts of the kinematic analysis of plane member systems, basic construction rules of geometrically unchangeable systems, and the computational degree of freedom of member systems will be discussed.

2.1 Several Concepts of Kinematic Analysis

2.1.1 Degree of Freedom and Constraint

To evaluate if a plane system is geometrically unchangeable is actually to distinguish whether there exist degrees of freedom of rigid motion in the system. The **degree of freedom** (DOF) refers to the number of independent parameters or coordinates used to determine the system's positions. In other words, DOF is the coordinate number that can be changed independently when the system moves.

Based on the theoretical mechanics, two coordinates x and y are required to determine the position of a mass point shown in figure 2.2a, thus a point has two DOFs in the plane. For a rigid body in the plane shown in figure 2.2b, a third independent coordinate φ is also required to determine its position in addition to x and y , so a rigid body has three DOFs in the plane. In the plane member system, when the member deformation is neglected, any member or geometrically unchangeable part composed of multiple members can be treated as a rigid body, which can be named a rigid disc. Therefore, a rigid disc also has three DOFs in the plane.

Generally, engineering structures are all geometrically unchangeable, and the DOF of the structure is zero. The DOF of the geometrically changeable system is always positive, and *vice versa*.

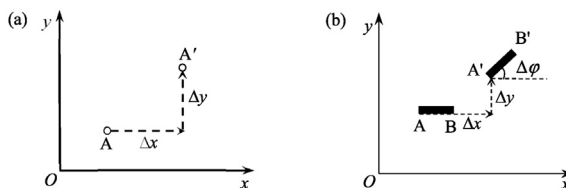


FIG. 2.2 – Degrees of freedom of (a) a mass point and (b) a rigid body in a plane.

When some restriction devices are applied on a point or a rigid disc, its DOF will decrease. The restriction device is named a **constraint**, and the constraint number equals the decreased number of DOF. In a plane system, the commonly used constraints among members or between members and supports are links,

hinges, and rigid joints. As shown in figure 2.3a, a link is used to constrain two rigid discs. Two independent rigid discs have 6 DOFs in the plane, but when they are constrained by a link, three coordinates are required to determine the position of rigid disc I and only two coordinates are needed to determine the position of rigid disc II, so the DOF of the system is 5. Therefore, one link is equivalent to 1 constraint, which can decrease 1 DOF. As shown in figure 2.3b, two rigid discs are constrained by one hinge. For this system, 3 coordinates are required to determine the position of rigid disc I and only 1 coordinate is needed to determine the position of rigid disc II, thus this system has 4 DOFs. As a result, 1 hinge is equivalent to 2 constraints, which can decrease 2 DOFs. Figure 2.3c demonstrates a system composed of two rigid discs, which are connected with 1 rigid joint, and the system turns into an extended rigid disc, so it has 3 DOFs. Accordingly, 1 rigid joint is equivalent to 3 constraints, which can decrease 3 DOFs.

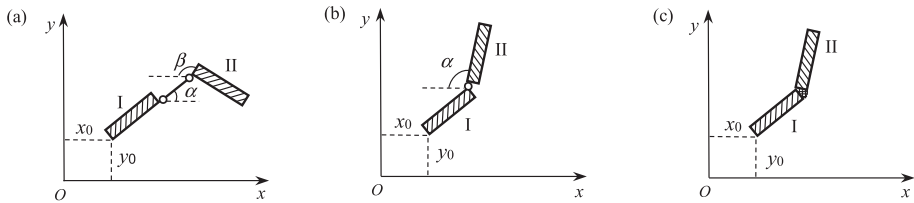


FIG. 2.3 – (a) Link, (b) hinge, and (c) rigid joint used to restrain two rigid discs.

In fact, constraints can be classified into simple constraints and complex constraints. A link that connects two points is called a simple link, and it is equivalent to 1 constraint. A link connecting three or more points can be named a complex link, as shown in figure 2.4a. Three points have 6 DOFs in the plane, and their DOF is reduced to 3 when they are constrained by a link. Thereby, a complex link connecting 3 points is equivalent to 3 constraints, *i.e.*, 3 simple links. By analogy, from the viewpoint of decreasing DOF, a complex link connecting l points is equivalent to $2l - 3$ simple links.

A hinge or rigid joint connecting two rigid discs is called a simple hinge or simple rigid joint, while it can be named a complex hinge or complex rigid joint respectively when it connects more than two rigid discs, as shown in figure 2.4b, c. In the same way, a complex hinge or a complex rigid joint which connects l rigid discs is equivalent to $l - 1$ simple hinges or simple rigid joints, *i.e.*, $2(l - 1)$ or $3(l - 1)$ constraints respectively.

However, it is noted that not all constraints can decrease the DOF of a system. When a constraint is added to a system, the DOF does not decrease, and then the constraint is called a **redundant constraint**. As shown in figure 2.5, point A has 2 DOFs in the plane, and it is connected with the foundation with two non-collinear links 1 and 2. Then, point A is completely fixed, and its DOF is 0. If one more link 3 is added, the DOF of the system will still be 0, so the link can be regarded as a redundant constraint. It should be noted that the number of redundant constraints

of a system is determined, while their locations are not unique and undetermined. In the system shown in figure 2.5, any link can be regarded as the redundant constraint of the other two links. Besides redundant constraints, other constraints are called **required constraints**, *i.e.*, the minimum number of constraints required to make a system geometrically unchangeable.

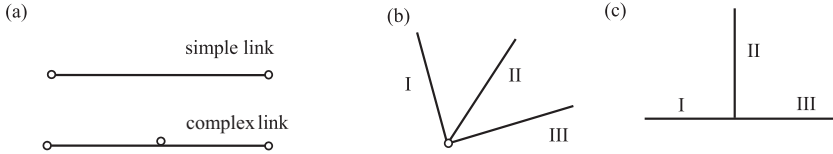


FIG. 2.4 – (a) Complex link, (b) complex hinge, and (c) complex rigid joint used to restrain multiple rigid discs.

Hence, to determine if a system is geometrically unchangeable, it is necessary to distinguish required constraints from redundant constraints. Only the required constraint has an effect on the DOF of a system, while the redundant constraint has no effect.

Two conditions are required for the construction of a geometrically unchangeable system with members, *i.e.*, enough constraints or connections and a suitable arrangement of them.

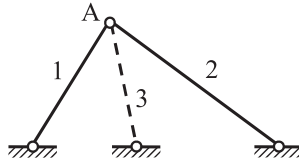


FIG. 2.5 – System with a redundant constraint.

2.1.2 Instantaneously Changeable System and Constantly Changeable System

Geometrically changeable systems can be classified into instantaneously changeable systems and constantly changeable systems. The instantaneously changeable system is a special case of the geometrically changeable system. As shown in figure 2.6a, point A is constrained on the foundation with two collinear links. Point A can move along the common tangent of circular arcs I and II. Actually, only 1 DOF (in the horizontal direction) of point A is reduced, and the DOF of the system is 1, so it is a geometrically changeable system. However, when point A has a tiny displacement δ along the movable direction, the two links become non-collinear, and the movements of point A along two directions can be constrained. Therefore, the DOF of the system is zero, and it turns into a geometrically unchangeable system. This kind of

system, which is originally geometrically changeable and will become geometrically unchangeable after a tiny movement, is called an **instantaneously changeable system**. Besides, a changeable system that can take large rigid body displacement is called a **constantly changeable system**, as shown in figure 2.6b. A constantly changeable system may have redundant constraints, but an instantaneously changeable system must have one or more redundant constraints. Even under a very small load, an instantaneously changeable system will have extremely large internal forces, which will lead to system destruction, and thus it cannot be used as an engineering structure.

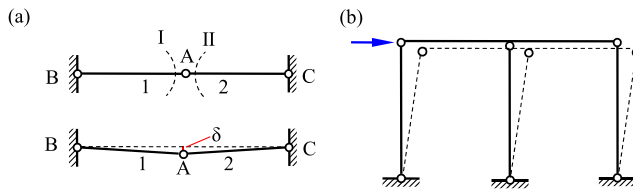


FIG. 2.6 – (a) Instantaneously changeable system and (b) constantly changeable system.

2.1.3 Instantaneous Hinge

As shown in figure 2.7, rigid disc I is attached to rigid disc II with two non-collinear and non-parallel links. Assuming rigid disc II is fixed, then rigid disc I has only two constraints, so its DOF is 1. When rigid disc I moves, link AB will rotate around B, so A will move along the tangent of AB. Similarly, C will move along the tangent of CD. Therefore, rigid disc I will have a small rotation around the intersection point O of AB and CD. Point O is the instantaneous rotation center of rigid discs I and II, and its effect is equivalent to connecting rigid discs I and II with a hinge at point O. Therefore, from the viewpoint of instantaneous tiny movement, the constraint effect of two links connecting two rigid discs is equivalent to a simple hinge on the intersection point. The location of the hinge changes with the movement of the links, hence it is called an **instantaneous hinge**. In addition, the hinge is not real, so it is also called a fictitious hinge.

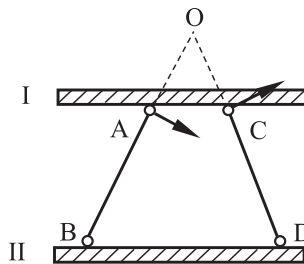


FIG. 2.7 – Instantaneous hinge.

There is a special case, namely, if two parallel links are used to restrain two rigid discs, as shown in figure 2.8a, then these two links are assumed to have an intersection at the infinity (∞) point. In other words, the equivalent instantaneous hinge of these two links is at the infinity point, which is named an infinity instantaneous hinge. Tiny rotation around the infinity instantaneous hinge degenerates into the translation along the perpendicular direction of these two links, as displayed in figure 2.8b, in which tiny displacements of points A and C are perpendicular to these two links.

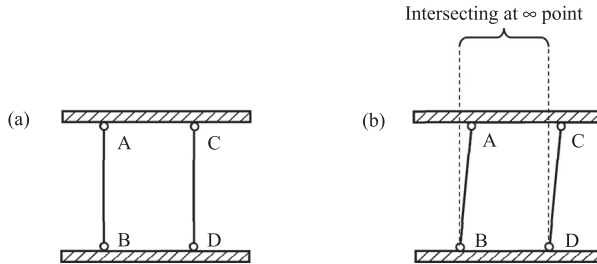


FIG. 2.8 – Rigid discs restrained with two parallel links.

When the concept of infinity instantaneous hinge is applied in the kinematic analysis, four conclusions about the ∞ point and ∞ line can be adopted:

- (1) Every direction has a ∞ point (the intersection of the parallel lines of the direction).
- (2) Different directions have different ∞ points.
- (3) All the ∞ points are on the same line, *i.e.*, the ∞ line.
- (4) All the finite points are not on the ∞ line.

These four conclusions are helpful to analyze the construction rules of the plane member system which is connected by multiple parallel links.

2.2 Basic Construction Rules of Plane Geometrically Unchangeable Systems

To analyze and determine if a plane member system is geometrically unchangeable, basic construction rules of the geometrically unchangeable system without redundant constraints need to be studied first. Basic rules of the geometrically unchangeable system constructed with points and rigid discs are discussed in this section, and some examples of the kinematic analysis are presented.

2.2.1 The Rule of Pin-Joined Member System

To establish a geometrically unchangeable system with a point and a rigid disc in the plane, if the rigid disc is fixed, only two non-collinear links are needed to fix the point on the rigid disc, as shown in figure 2.9, and then the system is geometrically

unchangeable without redundant constraints. In this context, these two non-collinear links are named as a **pin-joined member system**.

The rule of the pin-joined member system can be concluded as: a point and a rigid disc are connected with two links, the three hinges are non-collinear, and then the system is geometrically unchangeable. This is the connection way of a point and a rigid disc to establish a geometrically unchangeable system without redundant constraints.

Obviously, if a pin-joined member system is added to a rigid disc, then the system is still geometrically unchangeable with no redundant constraints. Hereby, the addition or deletion of a pin-joined member system does not change the geometric stability of the original system.

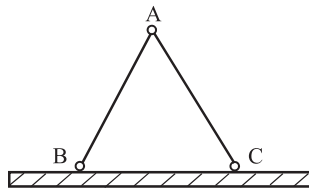


FIG. 2.9 – Pin-joined member system.

2.2.2 The Rule of Two Rigid Discs

To fix two rigid discs, if one of them is fixed, then another rigid disc has 3 DOFs relative to the fixed one, and three links are needed to fix this rigid disc in theory, as shown in figure 2.10a, b. The required precondition is that these three links are not concurrent or parallel with each other, because the constraint effect is equivalent to one hinge (or one fictitious hinge) in these two cases, which means only 2 DOFs can be reduced.

The rule of two rigid discs can be concluded as if two rigid discs are connected by three links, these three links are not concurrent, and then the system is geometrically unchangeable without redundant constraints.

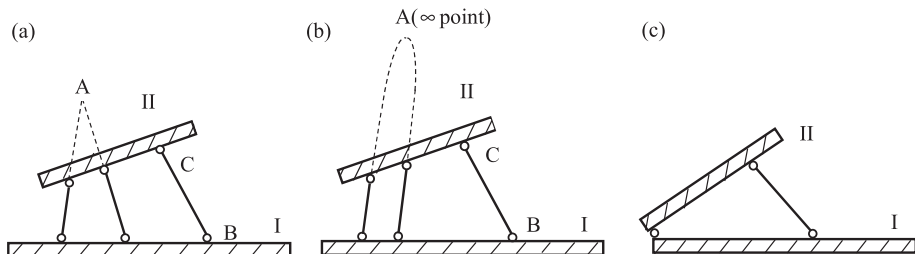


FIG. 2.10 – Two rigid discs fixed by three links (a), by two parallel links and another link (b), or by one link and one hinge (c).

As the constraint effect of two links is equivalent to a hinge, it is concluded that if two rigid discs are connected by a link and a hinge, three hinges are not collinear, and then the system is geometrically unchangeable without redundant constraints, as shown in figure 2.10c.

2.2.3 The Rule of Three Rigid Discs

To fix three rigid discs, if one of them is fixed, then the other two have 6 DOFs, and three hinges are needed to establish a geometrically unchangeable system in theory, as shown in figure 2.11a. The required precondition is that the three hinges should not be collinear.

The rule of three rigid discs can be concluded as: if three rigid discs are connected in pairs with three hinges, these three hinges are not collinear, and then the system is geometrically unchangeable without redundant constraints.

Hinges connecting the three rigid discs can be real hinges or fictitious hinges. In other words, any of the three hinges can be replaced by equivalent links, and the rule of three rigid discs is still applicable, as displayed in figure 2.11b.

The above paragraphs illuminate several basic rules to construct geometrically unchangeable systems without redundant constraints. These rules can be used to decide whether some simple plane member systems are geometrically unchangeable and whether the systems have redundant constraints, which are the specific applications in the kinematic analysis of these basic rules.

Before introducing the specific applications of these basic rules, their common points are first analyzed here. Actually, the rule of the pin-jointed member system, the rules of two rigid discs and three rigid discs are all based on the same law, *i.e.*, the geometric shape of a hinged triangle with given side lengths is uniquely determined. A hinged triangle is the simplest geometrically unchangeable system without redundant constraints. Therefore, the basic rule of the plane geometrically unchangeable system can be named the rule of the hinged triangle uniformly. The preconditions of these three basic rules include two links non-collinear, three links non-concurrent, and three hinges non-collinear, whose purposes are to guarantee the construction of the stable hinged triangle.

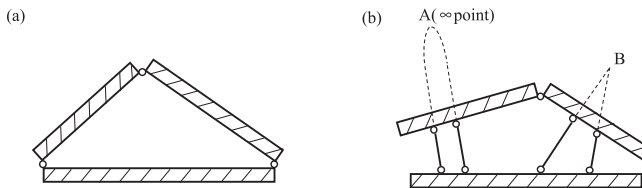


FIG. 2.11 – Three rigid discs fixed by three real or fictitious hinges.

To construct a geometrically unchangeable system without redundant constraints, its assembling processes can be divided into two categories:

(1) To perform the assembling from the foundation: the foundation is regarded as the basic rigid disc, and surrounding parts (a joint, a rigid disc, or two rigid discs)

are fixed on the basic rigid disc according to the basic rules to form an extended rigid disc. Then, the entire system is assembled one by one, from near to far rigid discs and from small to large rigid discs in terms of the basic construction rules. Figure 2.12 demonstrates an example of the assembling process.

The member system displayed in figure 2.12 is assembled from the foundation and through repeated applications of the basic construction rules, and different basic rules can be used to analyze the assembling process. With the rule of the pin-jointed member system, this system can be considered to be constructed from the foundation and with the additions of pin-jointed member systems C, D, and E in sequence. With the rule of two rigid discs, the foundation is regarded as the basic rigid disc, which composes an extended rigid disc with pin-jointed member system C, and the extended rigid disc is named rigid disc I. In addition, the hinged triangle BDE is a geometrically unchangeable system without redundant constraints, which consists of three links connected in pairs, and it is named rigid disc II. Rigid disc II is fixed on the rigid disc I through link CD and hinge B, and then the entire system is formed. In the same way, with the rule of three rigid discs, link CD can be regarded as rigid disc III, and then rigid disc II and III are fixed on the rigid disc I with three non-collinear hinges, and the final geometrically unchangeable system without redundant constraints is formed.

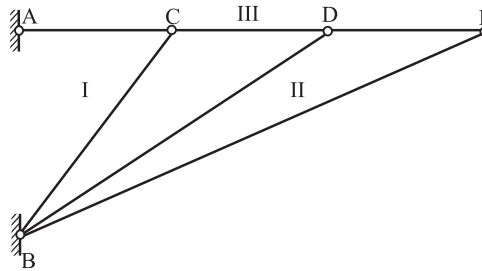


FIG. 2.12 – Plane member system with three pinned joints.

(2) To perform the assembling from the interior: firstly one or several rigid discs from the interior of the member system is chosen as the basic rigid discs, and then the surrounding parts are assembled based on the basic construction rules to obtain one or several extended basic rigid discs. Finally, the extended basic rigid discs are assembled with the foundation to form the entire system. Figure 2.13 illustrates an example of the assembling process.

Figure 2.13 displays a plane member system with curved members. Curved member AC and pin-jointed member system D form a geometrically unchangeable system without redundant constraints, which is named as rigid disc I. In the same way, curved member BC and pin-jointed member system E are named rigid disc II. These two rigid discs are connected with link DE and hinge C, and the three hinges are not collinear. Therefore, they form a geometrically unchangeable system based on the rule of two rigid discs, *i.e.*, an extended rigid disc. The extended rigid disc is

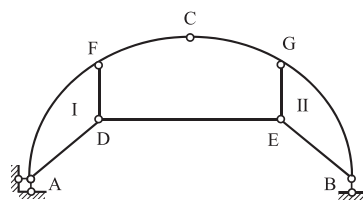


FIG. 2.13 – Plane member system with curved members.

connected with the foundation through three links that are not concurrent, and then the original system is geometrically unchangeable without redundant constraints with another application of the rule of two rigid discs.

The geometrically unchangeable system which does not depend on the foundation is called the interior unchangeable system, and otherwise, it is called the interior changeable system. From the above examples, some general conclusions can be obtained: (1) If the interior of the system and the foundation are connected with three links that are not concurrent, then the geometrically unchangeable characteristic of the system only depends on the interior. (2) If the interior unchangeable system and the foundation are connected with three links that are not concurrent, then it is still geometrically unchangeable. (3) If the interior changeable system is connected with the foundation in the same way, then it is still geometrically changeable.

From the above conclusions, an analysis skill can be summarized: a system which is formed by connecting the foundation with three non-concurrent links can be analyzed by disconnecting it with the foundation. In other words, for the plane member system the interior of which is connected with the foundation through three constraints (three links which are not concurrent), the kinematic analysis can begin from the interior of the system, and then consider the connections with the foundation.

Next, the kinematic analysis of some plane member systems will be performed with the three basic rules and two assembling processes, and the application skills are discussed.

Example 2.1. Perform the kinematic analysis for the plane system shown in figure 2.14a.

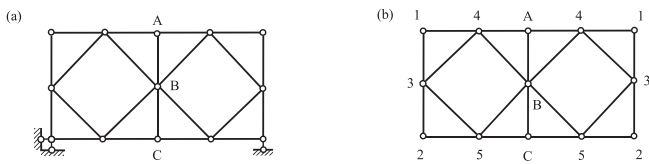


FIG. 2.14 – Member system for example 2.1.

Solution: The system is connected to the foundation through three links that are not concurrent, so the analysis can be performed by disconnecting the system from the foundation, as shown in figure 2.14b. This system is bilateral symmetry when deleting pin-jointed member systems 1, 2, 3, 4, and 5 in sequence, and then only links AB and BC are left. Obviously, this system is geometrically changeable.

Example 2.2. Perform kinematic analysis for the two systems shown in figure 2.15a, b.

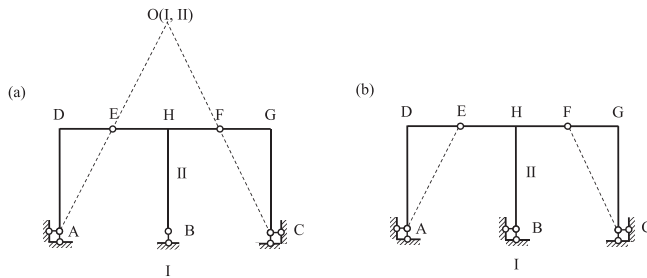


FIG. 2.15 – Member system for example 2.2.

Solution: The system shown in figure 2.15a is analyzed first. There are 5 constraints between the interior of the system and the foundation (more than three constraints), and the kinematic analysis cannot be performed by disconnecting the interior of the system from the foundation, so the analysis should begin from the foundation. Points A and C are connected with the foundation with two links respectively to form an extended rigid disc, which can be named rigid disc I. The T-shape member BEF can be named rigid disc II. Rigid discs I and II are connected through the link at point B and the fold members ADE and CGF. The constraint effects of ADE and CGF are equivalent with links AE and CF respectively. The system is symmetry about axis BH, so these three links are certainly concurrent at point O (I, II), and this point is the instantaneous rotation center of rigid disc II relative to rigid disc I. When rigid disc II rotates a small angle, the three links will be non-concurrent, and then the system becomes geometrically unchangeable without redundant constraints. Therefore, the system is an instantaneously changeable system.

Then, the system shown in figure 2.15b is analyzed. In the same way, points A, B, C and the foundation can be regarded to form an extended rigid disc I and BEF is rigid disc II. These rigid discs are connected with hinge B and fold members ADE and CGF, and the equivalent members of the fold members do not go through hinge B. Therefore, the system is geometrically unchangeable, but it has one redundant constraint.

Example 2.3. Perform the kinematic analysis for the system shown in figure 2.16a.

Solution: Figure 2.16a demonstrates a plane member system formed by simple members connected with hinges. When performing the kinematic analysis, each member can be regarded as a rigid disc, or as a link constraint among rigid discs. Then, according to the choice, there are different methods for the kinematic analysis. For this example, two methods are available.

With the first method, three members shown in figure 2.16b can be regarded as rigid discs I, II, and III, and then other members are regarded as the constraints among rigid discs. Any two rigid discs are connected with two links, and the constraint effect is equivalent to one fictitious hinge. According to the geometric symmetry of the system, it is obvious that these three fictitious hinges are collinear, so the interior of the system is instantaneously changeable.

With the second method, three members shown in figure 2.16c can be regarded as rigid discs and other members are regarded as the constraints, and then any two rigid discs are connected with two parallel links, *i.e.*, the equivalent fictitious hinge is at the infinity point. According to the conclusion of the infinity point in the projective geometry, all the infinity points are collinear. Therefore, the interior of the system is instantaneously changeable.

It can be seen from the above examples that the basic construction rules of a geometrically unchangeable system can be used to perform the kinematic analysis. In general, there are two analysis methodologies: addition and deletion. For some systems, the addition analysis methodology can be used. Beginning from one or several rigid discs, the whole system is formed step by step according to the rules. For other systems, the deletion analysis methodology can be used. Deleting the pin-jointed member system from the system gradually, or disconnecting the system from the foundation, aims to reduce the analysis complexity. When analyzing different plane systems with the three basic construction rules, the key problem is which parts should be chosen as rigid discs and which parts should be chosen as constraints.

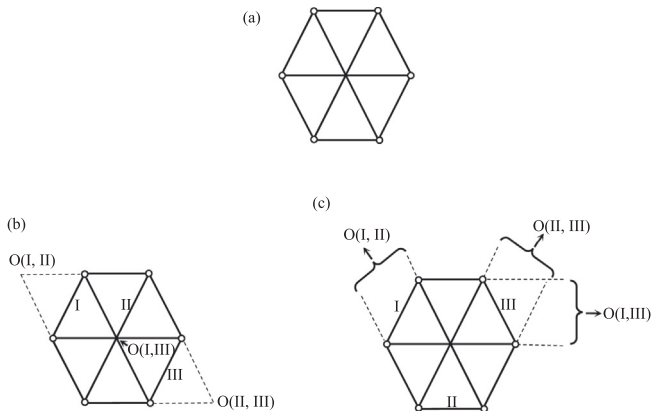


FIG. 2.16 – Member system for example 2.3.

2.3 Computational Degree of Freedom of Plane Member Systems

Using the rule of the hinged triangle, the kinematic analysis can be performed for general plane member systems, and the following two questions can be answered, and some conclusions can be concluded:

- (1) Whether the system is geometrically changeable? What is the number S of DOF?
- (2) Whether the system has redundant constraints? What is the number n of redundant constraints?

If $S > 0$, then the system is geometrically changeable and it cannot be used as a structure. It is a constantly changeable system when $n = 0$, and it is an instantaneously changeable system or a constantly changeable system when $n > 0$. If $S = 0$, then the system is geometrically unchangeable. It is a statically determinate structure when $n = 0$, and it is statically indeterminate when $n > 0$.

However, for some complex member systems, it is very difficult to perform the kinematic analysis based on the rule of the hinged triangle, and then how to determine the geometric stability of the system and get S and n ? A system is composed of parts (points and rigid discs) and constraints. The DOF S of the system can be solved by subtracting the number of the required constraints from the sum of the DOF of all the parts when they all are independent. However, for complex systems, the required constraints cannot be determined directly in general. Therefore, the **computational degree of freedom** W is introduced, and then some useful conclusions about S and n are obtained based on W .

The computational degree of freedom of the system is defined as the total DOF a of all the parts minus the total constraint number d

$$W = a - d \quad (2.1)$$

According to the above equation, the solution of W only needs to determine the total number of the DOF and the total number of constraints, which are relatively easy. Next, the relation between the computational degree of freedom and the DOF is further discussed. According to the definition of the DOF

$$S = a - c \quad (2.2)$$

in which c is the number of the required constraints, and the number n of redundant constraints is expressed as

$$n = d - c \quad (2.3)$$

Based on the above equations, it can be concluded that

$$S - W = n \quad (2.4)$$

Therefore, the difference between the DOF S of the system and the computational degree of freedom W equals the number n of redundant constraints. In addition, S and n are always larger than or equal to 0, so

$$S \geq W, n \geq -W \quad (2.5)$$

The above equations show the relations among S , W , and n . W is the lower bound of S , and $-W$ is the lower bound of the required constraint number n . According to these relations and the value of W , which may be positive, negative, or zero, some qualitative conclusions can be get:

- (1) $W > 0$, then $S > 0$, and the system is geometrically changeable.
- (2) $W = 0$, then $S = n$. If the redundant constraints do not exist, *i.e.*, $n = 0$, then the system is geometrically unchangeable, and it is statically determinate. If there are redundant constraints, *i.e.*, $n > 0$, then the system is geometrically changeable.
- (3) $W < 0$, then $n > 0$, and the system has redundant constraints. If the system is geometrically unchangeable, then it is statically indeterminate.

Then, $W > 0$ is the sufficient condition of a geometrically changeable system, which implies that the total constraint number of the system is less than the total number of DOF. $W \leq 0$ is the necessary condition of a geometrically unchangeable system, which means the total constraint number of the system is not less than the total number of DOF. As long as the connection method of constraints is proper, the system is geometrically unchangeable.

When solving the computational degree of freedom W of the system, a part of the system should be regarded as parts, and the other part should be regarded as constraints, in which the complex constraints should be converted into equivalent simple constraints. A plane member system can be regarded as being composed of two parts: hinges and rigid discs, so three systems can be adopted to solve W , and thus three methods can be used. The first method regards rigid discs as parts, and the system is named a rigid disc system. The second method regards hinges as parts, and then the system is named a hinged system. The last method regards the system as a combination of hinge and rigid disc parts. The methods to solve W of these three systems are introduced as follows.

2.3.1 Computational Degree of Freedom of Rigid Disc System

As mentioned above, a rigid disc system takes rigid discs as parts, and the system can be regarded as being composed of rigid discs which are constrained with rigid joints, hinges, and links. Assuming the numbers of rigid discs, simple rigid joints, simple hinges, and simple links are m , g , h , and b , respectively. According to equation (2.1), the computational degree of freedom W of the rigid disc system is written by

$$W = 3m - (3g + 2h + b) \quad (2.6)$$

It should be noted that the constraint number in the equation refers to the number of simple constraints, so complex constraints in the system need to be turned into equivalent simple constraints. In addition, the interior of rigid discs

should have no redundant constraints, otherwise, the redundant constraints should be counted in the total constraint number.

Example 2.4. Solve the computational degree of freedom W of the plane member system shown in figure 2.17.

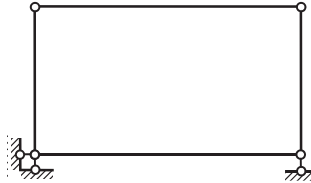


FIG. 2.17 – Member system for example 2.4.

Solution: Members in the system are regarded as rigid discs, and the number is 4. Rigid discs are constrained with hinges, the number of which is 4. In addition, three links are used to fix the rigid discs on the foundation. According to equation (2.6), $m = 4$, $h = 4$, $b = 3$. So

$$W = 3m - 2h - b = 3 \times 4 - 2 \times 4 - 3 = 1$$

Then the DOF S of the system is larger than 0, and the system is geometrically changeable.

Example 2.5. Solve the computational degree of freedom W of the system shown in figure 2.18.

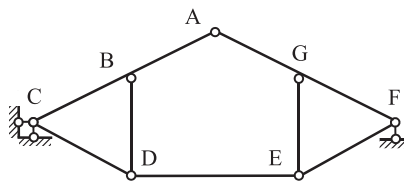


FIG. 2.18 – Member system for example 2.5.

Solution: Members in the system are regarded as rigid discs, the number of which is 7, and they are constrained with hinges. The difference with the above example is that the two hinges on the lower part of the system connect three rigid discs respectively, so these hinges are complex hinges, and they should be converted to equivalent simple hinges. According to section 2.1.1, a complex hinge or a complex rigid joint connecting n rigid discs is equivalent to $n - 1$ simple hinges or simple rigid joints, so each of these two complex hinges is equivalent to two simple hinges respectively. As a result, the system in figure 2.18 has 9 simple hinges in total.

In addition, the system also has three link constraints. According to equation (2.6), then

$$W = 3m - 2h - b = 3 \times 7 - 2 \times 9 - 3 = 0$$

There are no redundant constraints in the system, so the system is geometrically unchangeable.

Example 2.6. Solve the computational degree of freedom W of the system shown in figure 2.19a.

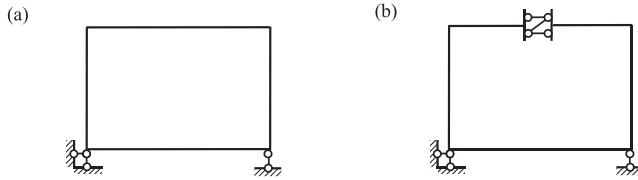


FIG. 2.19 – Member system for example 2.6.

Solution: Figure 2.19a shows a rectangle rigid disc that is fixed on the foundation with three links. Thus, does the computational degree of freedom of the system equal to zero? The answer is negative. The rectangle rigid disc is a system including redundant constraints, so it should be converted to a rigid disc without redundant constraints as well as additional constraints. In this way, the rectangle disc can be cut off from the middle of a member, so it will have no redundant constraints. Thereafter, the left and right cross sections at the cut point are constrained with three links as shown in figure 2.19b, and their relative horizontal, vertical, and angular displacements are restricted, so the system is equivalent to the original system. Finally, the computational degree of freedom can be calculated according to equation (2.6),

$$W = 3m - b = 3 \times 1 - 6 = -3$$

2.3.2 Computational Degree of Freedom of the Hinged System

The hinged system can be regarded as being formed by hinges constrained by links. Assuming the hinge number of the hinged system is j , the number of links is b , then the computational degree of freedom W is expressed as

$$W = 2j - b \quad (2.7)$$

Similarly, the link constraint in the above equation refers to simple links, and if there are complex links in the system, the complex links should be converted to equivalent simple links.

Example 2.7. Solve the computational degree of freedom W of the system shown in figure 2.17.

Solution: The system shown in figure 2.17 has been solved in example 2.4 as a rigid disc system. Here, the system is viewed as a hinged system. There are 4 hinges in the system and 4 links among hinges. Meanwhile, the system is fixed on the foundation with 3 links, so there are 7 link constraints. According to equation (2.7), the computational degree of freedom is

$$W = 2j - b = 2 \times 4 - 7 = 1$$

The result is the same as that of example 2.4.

Example 2.8. Calculate the computational degree of freedom W of the system shown in figure 2.20.

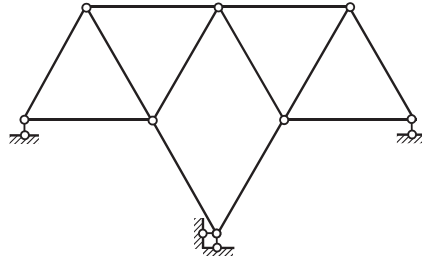


FIG. 2.20 – Member system for example 2.8.

Solution: The system in figure 2.20 has 8 pinned joints, and there are 12 link constraints among the joints. Meanwhile, the system is fixed on the foundation with 4 links, and it totally has 16 link constraints, so

$$W = 2j - b = 2 \times 8 - 6 = 0$$

Example 2.9. Solve the computational degree of freedom W of the system shown in figure 2.18.

Solution: There are several pinned joints in the system displayed in figure 2.18, so it is suitable for being analyzed as the hinged system. The system totally has 7 hinges, and there are 7 links among the hinges. However, it is worth noting that there are two links in the upper part of the system which connects 3 hinges respectively, so they are complex links. According to section 2.1.1, a complex link connecting n points is equivalent to $2n - 3$ simple links, so the complex link in this system is equivalent to 3 simple links. Therefore, there are a total of $3 + 3 + 5$ simple links among hinges, and the system is fixed on the foundation with 3 links. Accordingly, the computational degree of freedom of the system is

$$W = 2j - b = 2 \times 7 - 14 = 0$$

The result is consistent with that of example 2.5.

2.3.3 Computational Degree of Freedom of the Mixed System

For some systems, whether it is solved as the rigid disc system or hinged system, there will always be complex constraints, and the complex constraints should be converted to equivalent simple constraints, which further increase the computational complexity. For avoiding this situation, when calculating the computational degree of freedom of some systems, one part of the system can be regarded as the rigid disc system, and the other part can be regarded as the hinged system. Accordingly, the computational degree of freedom of a mixed system is formulated as

$$W = 3m + 2j - (3g + 2h + b) \quad (2.8)$$

in which m is the number of rigid discs in the rigid disc system, j is the number of hinges in the hinged system, g and h are the numbers of simple rigid joints and simple hinges in the rigid disc system, and b is the number of simple links in the system.

The system shown in figure 2.18 is still used as an example here, and it includes complex hinges and complex links, so there will be complex constraints no matter whether the system is treated as a rigid disc system or a hinged system. Thus, the system can be treated as a mixed system. Members connecting several hinges are regarded as rigid disc parts, and hinges connecting several members are regarded as hinge parts. Therefore, members AC, and AF are regarded as rigid discs, and the number of rigid joints and hinges between them is 0 and 1, respectively. The other part of the system is regarded as the hinged system, and there are 2 hinges (D and E), 5 links among the hinges, and rigid discs. In addition, there are 3 links between the system and the foundation. Thus, the computational degree of freedom of the mixed system is

$$W = 3m + 2j - (3g + 2h + b) = 3 \times 2 + 2 \times 2 - (2 \times 1 + 8) = 0$$

The result is consistent with that of examples 2.5 and 2.9.

Example 2.10. Solve the computational degree of freedom W of the system shown in figure 2.21.

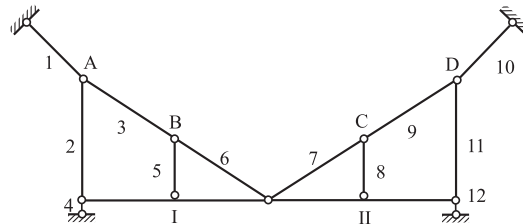


FIG. 2.21 – Member system for example 2.10.

Solution: The system shown in figure 2.21 also includes complex links and complex hinges. In the same way, members connecting several hinges are regarded as the rigid disc system, so the number of the rigid discs is 2, and the number of hinges between

rigid discs I and II is 1. The other part is regarded as the hinged system, and there are 4 hinges A, B, C, and D. The number of link constraints in the system is 12. Hence, the system can be regarded as the mixed one, and the corresponding computational degree of freedom is easy to be calculated as

$$W = 3m + 2j - (3g + 2h + b) = 3 \times 2 + 2 \times 4 - (2 \times 1 + 12) = 0$$

As indicated in the above examples, the system can be treated as the rigid disc system, hinged system, or mixed system as needed, which is helpful to solve the computational degree of freedom W quickly and accurately, and then to perform the kinematic analysis.

2.4 Geometrical Stability and Static Determinacy of Systems

The kinematic analysis can not only determine the geometrical stability but also can illustrate the static determinacy of systems. **Static determinacy** refers to that if all the reactions and the internal forces of a system under arbitrary loads can be determined by the static equilibrium conditions. In this section, the relation between the geometrical stability and the statically determinacy of systems will be discussed.

Generally, constantly changeable or unstable systems cannot keep equilibrium under arbitrary loads, *i.e.*, the equilibrium condition does not hold, so the equilibrium equation has no solution. A constantly changeable system cannot be used as a structure.

The instantaneously changeable system has been analyzed in section 2.1, and it will have infinite internal forces even under a very small load, and the equilibrium equation also has no solution. Therefore, the instantaneously changeable system cannot be used as a structure either, and the application of a structure close to the instantaneously changeable system should be avoided in structural design.

If the system is geometrically unchangeable, then it will maintain equilibrium under arbitrary loads, so the equilibrium equation certainly has a certain solution. However, for geometrically unchangeable systems without or with redundant constraints, their solutions are different.

The simply supported beam shown in figure 2.22a is a geometrically unchangeable system without redundant constraints, and it has three unknown support reactions. Beam AB (rigid disc) can be regarded as a free body, and then three equilibrium equations can be formed to uniquely determine the support reactions. Any cross-section of beam AB can be determined with the section method after the support reactions are solved. Therefore, the system is statically determined. A geometrically unchangeable system without redundant constraints can be named a statically determinate structure. The system shown in figure 2.22b is added a link based on the geometrically unchangeable system in figure 2.22a, and it is geometrically unchangeable with redundant constraints. When beam AB is regarded as a free body, the number of unknown support reactions is 4, which is bigger than

the number of the equilibrium equations. Thus, there are infinite sets of solutions for equilibrium equations, or in other words, the solution cannot be uniquely determined with the equilibrium equations. Therefore, a geometrically unchangeable system with redundant constraints is statically indeterminate, which can be named a statically indeterminate structure. The support reactions and internal forces of a statically indeterminate structure can be solved in conjunction with the deformation compatibility condition.

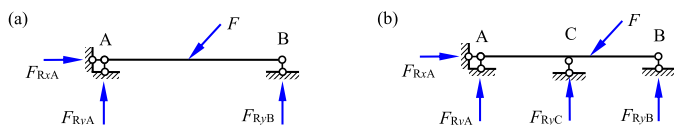


FIG. 2.22 – (a) Statically determinate system and (b) statically indeterminate system.

In summary, only the geometrically unchangeable system without redundant constraints is statically determinate. In other words, the geometrical stability of a statically determinate structure is geometrically unchangeable and has no redundant constraints. A system has redundant constraints on the basis of a statically determinate structure is a statically indeterminate structure. In this way, a structure can be determined as statically determinate or statically indeterminate through the study of the kinematic analysis of member systems in this chapter.

Problems

2.1 Perform the kinematic analysis of the systems shown in the following figures.

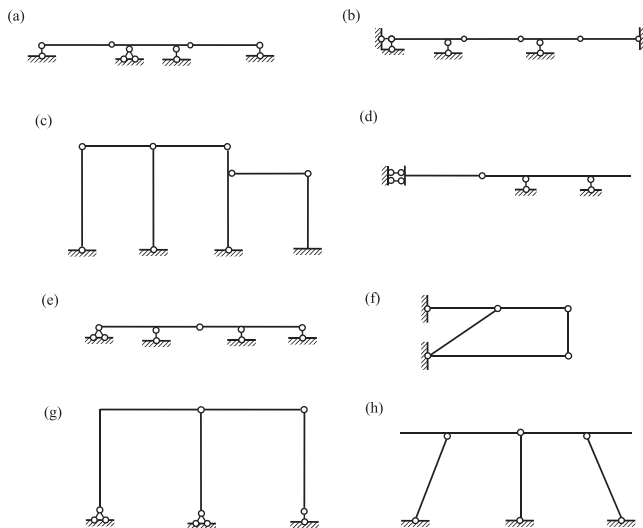


FIG. P2.1

2.2 Perform the kinematic analysis of the systems illustrated in the following figures.

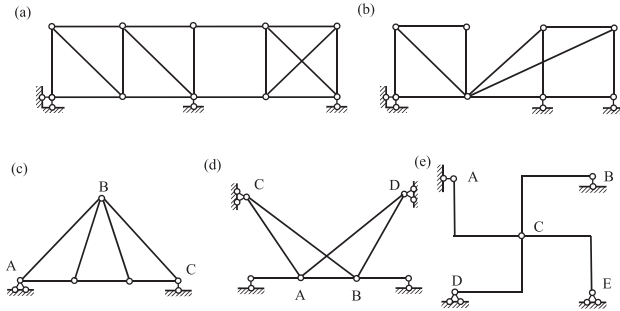


FIG. P2.2

2.3 Perform the kinematic analysis of the systems demonstrated in the following figures.

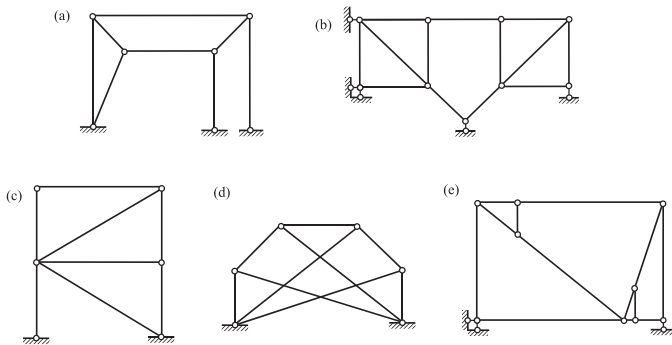


FIG. P2.3

2.4 Perform the kinematic analysis of the system shown in the following figures.

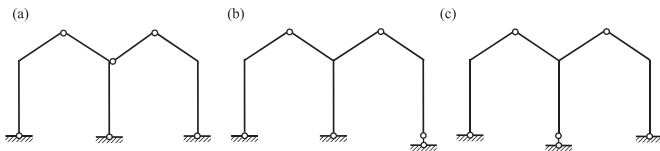


FIG. P2.4

2.5 Perform the kinematic analysis of the systems displayed in the following figures.

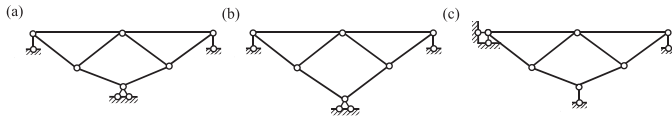


FIG. P2.5

2.6 Solve the computational degree of freedom W of the systems exhibited in the following figures.

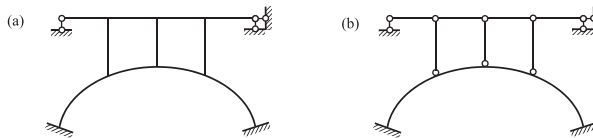


FIG. P2.6

2.7 Solve the computational degree of freedom W of the systems shown in the following figures.

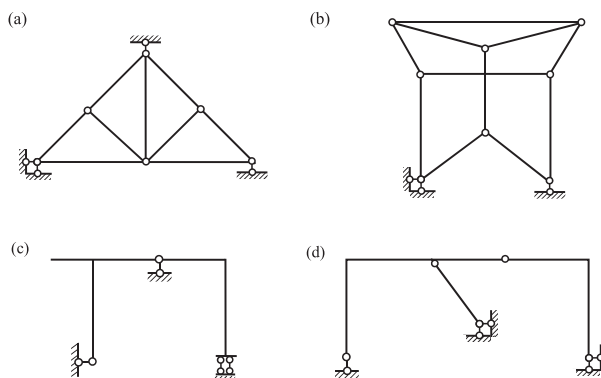


FIG. P2.7

Chapter 3

Analysis of Statically Determinate Structures

For **statically determinate structures**, all the reactions and internal forces can be determined by solving the static equilibrium conditions. Statically determinate structures are widely used in practical engineering applications, because of various advantages, *e.g.*, the force analysis is simple, the force state of the structure is not influenced by the material, no redundant members are needed for engineering constructions, and the material and cost saving, etc. Figure 3.1 shows several typical structures, which are a train axle, a bus shelter, a railway bridge, and a crane bridge in the plant, respectively, and their computational diagrams can be treated as statically determinate structures for the force analysis through approximations and simplifications.



FIG. 3.1 – Typical statically determinate structures.

According to the feature of geometrical construction, typical plane statically determinate structures can be categorized as single-span statically determinate beams, multi-span statically determinate beams, statically determinate plane trusses, statically determinate plane frames, composite structures, three-hinged arches, etc. Although the force analysis of different statically determinate structures can be addressed by establishing equilibrium equations of the selected free body, their force analysis has different characteristics due to different features of the geometrical construction. By grasping these characteristics, the force analysis for various statically determinate structures can be performed quickly and accurately. In this chapter, the relation between the geometrical construction and the force analysis, and the general solution order of internal forces are discussed for different statically determinate structures. The content includes the solution of support reactions and internal forces, drawing of internal force diagrams, analysis of mechanical performances, and so on.

3.1 Single-Span Statically Determinate Beams

Beams are frequently used in engineering structures, and they mainly generate bending internal forces under loads, and their force analysis is the foundation of the mechanical analysis of other structures. In the mechanics of materials, definitions of the internal forces of the beam, relations among internal forces, and corresponding solution methods have been studied. In this section, the internal forces of a single bending member and their solution are reviewed and supplemented with the single-span statically determinate beam as an example. This section is the foundation of the structural analysis in the following sections because complex member structures can be split into multiple single members to perform the force analysis and hence understand its whole mechanical performance.

3.1.1 *Single-Span Statically Determinate Beams and Internal Forces*

Most single-span statically determinate beams can be divided into three categories, and their computational diagrams are shown in figure 3.2.

- (1) Simply supported beam. One end of the beam is constrained with a pin support, and the other end is constrained with a roller support, as displayed in figure 3.2a.
- (2) Cantilever beam. One end of the beam is fixed, and the other end is free, as shown in figure 3.2b.
- (3) Simply supported extended beam. The beam is simply supported, but one or two ends of the beam extend out of the supports, as shown in figure 3.2c, d.

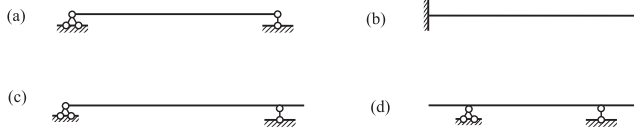


FIG. 3.2 – Single-span statically determinate beams.

The above single-span statically determinate beams all have three support reactions, and only the vertical reactions arise if the beam only has vertical loads. If the whole beam is treated as a free body, with the 3 equilibrium equations of a general planar force system:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$$

in which M is the bending moment of all the external forces to an arbitrary point in the plane of the force system, then all 3 reactions can be solved.

If a structure is in the equilibrium state under a force system, then any part of the structure is in equilibrium. Therefore, if the structure is split along the section in which the internal forces need to be solved, the part at any side of the section can be treated as a free body, and the equilibrium condition is utilized to calculate the internal forces. This basic force analysis method is called the **method of sections**.

The simply supported beam shown in figure 3.3a is truncated at the middle section of the beam, and the left part of the beam is treated as a free body. Then there are three internal forces in this section: the axial force F_N , the shear F_S , and the moment M , as shown in figure 3.3b.

The axial force F_N is the stress resultant on the section along the axial direction of the member, which numerically equals the algebraic sum of the projection along this direction of all the external forces on the free body, and tensile force is specified to be positive. The shear F_S is the stress resultant at the section along the normal direction of the member axis, which numerically equals the algebraic sum of the projection along this direction of all the external forces on the free body, and shear rotating clockwise around the free body is considered to be positive. The bending moment M is the moment of stress at the section with respect to the centroid, which numerically equals the algebraic sum of the moment of all the external forces on the free body with respect to the centroid of the section and the bending moment which makes the lower fibers of the beam at the section in tension is specified to be positive.



FIG. 3.3 – (a) Simply supported beam, (b) three internal forces on the section.

In structural mechanics, the positive and negative signs should be labeled on the axial force and shear diagrams of the beam. However, for the convenience of the structural analysis and design, moment diagrams are often drawn on the tensile side of the beam, without labeling the signs, and it is different from the convention of some textbooks on mechanics of materials in which bending moments are marked on the compressive side of the beam.

3.1.2 Relations Between Loads and Internal Forces

(1) Differential relations between loads and internal forces

In a straight beam with continuously distributed loads, a differential element is treated as a free body, the length of which is dx , as shown in figure 3.4, where q is the intensity of the vertical distributed load, F_S and M are the shear and the moment of the left section, $F_S + dF_S$ and $M + dM$ are the shear and moment of the right section. According to the equilibrium conditions of the differential element, three differential equations for the relations between loads and internal forces can be derived:

$$\frac{dF_S}{dx} = -q \quad (3.1)$$

$$\frac{dM}{dx} = F_S \quad (3.2)$$

$$\frac{d^2 M}{dx^2} = -q \quad (3.3)$$

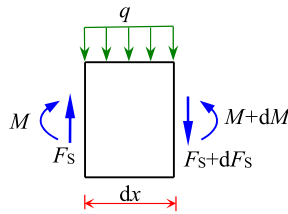


FIG. 3.4 – Differential element in a straight beam with continuously distributed loads.

These differential equations are equilibrium equations of the differential element, which indicate the relations between the internal forces and continuously distributed loads of the beam.

(2) Incremental relations between loads and internal forces

A differential element in a straight beam with a concentrated load is treated as a free body, as shown in figure 3.5, in which F_p is a vertical concentrated force, and M_0 is a clockwise couple moment. According to the equilibrium conditions, two incremental relations between the concentrated loads and internal forces can be achieved:

$$\Delta F_S = -F_p \quad (3.4)$$

$$\Delta M = M_0 \quad (3.5)$$

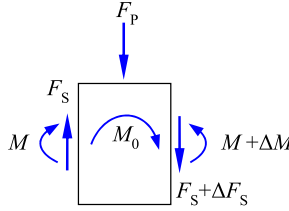


FIG. 3.5 – Differential element in a straight beam with a concentrated load.

(3) Integral relations between loads and internal forces

Segment AB is extracted from a straight beam with arbitrarily distributed loads, as shown in figure 3.6. Integration with equations (3.1) and (3.2) yields:

$$F_{SB} = F_{SA} - \int_{x_A}^{x_B} q(x) dx \quad (3.6)$$

$$M_B = M_A + \int_{x_A}^{x_B} F_S(x) dx \quad (3.7)$$

As indicated from the above two equations, the shear of end B equals the shear of end A subtracting the area of the distributed load $q(x)$, and the moment of end B equals that of end A adding the area of the shear diagram.

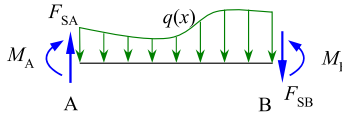


FIG. 3.6 – Segment AB with arbitrary distributed loads.

According to the above relations between loads and internal forces, some deductions can be further concluded:

- (1) For a member segment without vertical loads, its shear is constant, and its shear diagram is a straight line parallel to the member. Its moment diagram is an inclined straight line, the slope of which numerically equals the shear of the member.
- (2) For a member with vertical uniformly distributed loads, its shear diagram is an inclined straight line, the slope of which numerically equals the intensity of the uniformly distributed loads, but the signs are opposite. The moment diagram is a quadratic parabola.
- (3) When there is a vertical concentrated force or couple moment on the member, the shear or moment will have an abrupt change, and the variation equals the value of the load. The moment diagram will have a cusp at the section where the concentrated force is applied, and the direction of the cusp is identical to that of the concentrated force. The couple moment has no influence on the shear.
- (4) At the section where the shear is zero, the tangential direction of the moment diagram is parallel to the member, and the moment has an extremum.

With the above relations and deductions, the internal force diagrams can be quickly drawn and verified based on the external forces and reactions of the bending member.

3.1.3 Method of Segmental Superposition

The drawing of moment diagrams for the straight members in a structure can be simplified with the **method of segmental superposition**, which is helpful to draw the moment diagram rapidly and accurately.

Herein, the drawing methods of moment diagrams for arbitrary straight members are discussed. Figure 3.7a shows a simply supported beam with complex loads, which are different in different segments. For drawing the moment diagram of the whole beam, the simply supported beam can be divided into multiple segments, and each segment is equivalent to a simply supported beam subjected to the couple moments at the beam ends and transverse loads. The moment diagrams can be calculated separately for the couple moments at the beam ends and transverse loads, and they are superimposed to obtain the moment diagram of the equivalent simply supported beam. Finally, the moment diagrams of multiple segments are joined, and then the moment diagram of the whole beam is obtained.

Segment AB of the simply supported beam is used as an example here to explain the principle of the method of segmental superposition. Figure 3.7b shows the free body of segment AB, and it is under moments M_A and M_B , axial forces F_{NA} and F_{NB} , shears F_{SA} and F_{SB} on the beam ends besides the uniformly distributed load q . For explaining the characteristics of the moment diagram of segment AB, it is compared with the simply supported beam shown in figure 3.7c. Assuming the simply supported beam is under the same uniformly distributed load q and the same couple moments M_A and M_B at the beam ends, the vertical support reactions are labeled as F_{yA}^0 and F_{yB}^0 . Solving F_{SA} , F_{SB} , F_{yA}^0 and F_{yB}^0 shown in figure 3.7b, c with the equilibrium equations respectively, it is obvious that $F_{SA} = F_{yA}^0$, $F_{SB} = F_{yB}^0$,

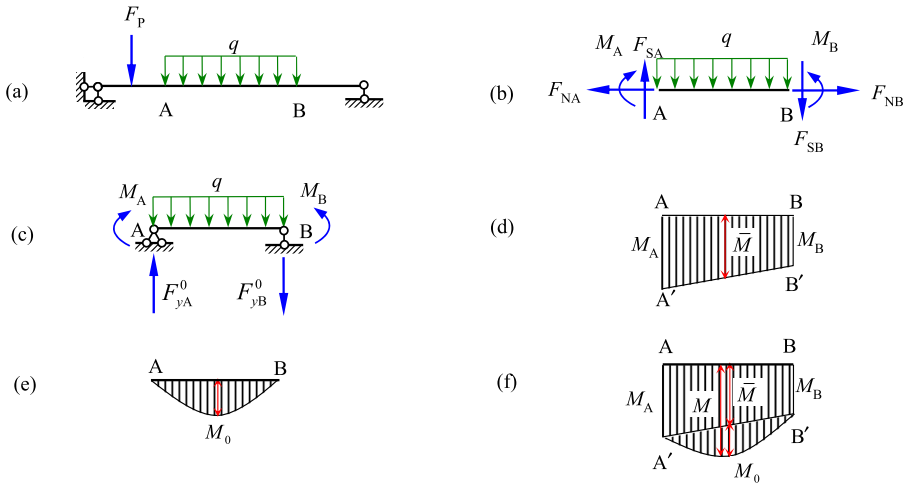


FIG. 3.7 – Force analysis of a simply supported beam with method of segmental superposition.

so the moment diagrams of figure 3.7b, c are the same. Therefore, the drawing of the moment diagram of an arbitrary straight member segment can be formulated as the problem of drawing the moment diagram of the equivalent simply supported beam. Then, the drawing method of the moment diagram of the equivalent simply supported beam shown in figure 3.7c will be discussed. The loads on the simply supported beam can be divided into two parts: the load q within the span and the couple moments M_A and M_B at the beam ends. According to the superposition principles: when there are multiple loads on a beam at the same time and under small deformation and linear elastic material conditions, the internal forces at an arbitrary section are equal to the algebraic sum of the internal forces when the loads act independently. With the couple moments at the beam ends acting alone, the moment diagram ($\bar{M}$ diagram) is a straight line as shown in figure 3.7d. With load q within the span acting alone, the moment diagram (M^0 diagram) is shown in figure 3.7e. Superposing the $\bar{M}$ and M^0 diagrams, the total moment diagram (M diagram) is obtained, as shown in figure 3.7f. It should be noticed that the overlying of moment diagrams is the superposition of longitudinal coordinates, which is not the simple splicing of diagrams. The relation of the three longitudinal coordinates $\bar{M}$, M^0 and M shown in figure 3.7f satisfies:

$$\bar{M}(x) + M^0(x) = M(x)$$

Example 3.1. Draw the internal force diagram of the simply supported beam shown in figure 3.8a.

Solution: Firstly, solve the whole equilibrium equations of the simply supported beam to get the support reactions:

$$F_{yA} = 5 \text{ kN} \cdot \text{m}(\uparrow)$$

$$F_{yH} = 3 \text{ kN} \cdot \text{m}(\uparrow)$$

Draw the moment diagram of the simply supported beam using the method of segmental superposition. Firstly, the simply supported beam AH is divided into three segments: AC, CE, and EH, and certainly AC or EH can be further divided into two segments. Segments AC, CE, and EH can be formulated as the simply supported beams shown in figure 3.8b–d.

Then solve the couple moments at the ends of the three simply supported beams, respectively. In figure 3.8b, end A is a pin support in the original simply supported beam, so the moment $M_A = 0$. The couple moment at end C equals the bending moment of the corresponding section in the original structure. Segment AC of the original structure is treated as a free body, and its equilibrium equations can be established, then

$$M_C = F_{yA} \times 2 \text{ m} - F_P \times 1 \text{ m} = 6 \text{ kN} \cdot \text{m}$$

In the same way, the couple moments at the ends of the equivalent simply supported beams CE and EH can be obtained

$$M_H = 0$$

$$M_E = F_{yH} \times 2 \text{ m} - M = 4 \text{ kN} \cdot \text{m}$$

With the couple moments at the beam ends acting alone, the moment diagrams of the equivalent simply supported beams are shown in figure 3.8e–g. With the loads within the span acting alone, the moment diagrams are shown in figure 3.8h–j. Then, the total moment diagram of the equivalent simply supported beam is the superposition of two-moment diagrams, as shown in figure 3.8k–m. Finally, the moment diagram of the original simply supported beam can be obtained by joining the moment diagrams of the equivalent simply supported beams of AC, CE, and EH, as shown in figure 3.8n.

Based on the method of segmental superposition, the general procedure to draw the moment diagram of a beam is summarized as follows.

- (1) Segmentation. Choose the discontinuous points of the external forces, such as the acting points of concentrated loads, the acting points of the couple moments, and the start and end points of distributed loads, as the control sections to divide the structure into multiple segments.
- (2) Moment determination of control points. Solve all the bending moments of the control sections.
- (3) Connecting the line of moment diagram. The lines for connecting the moments of the control sections are also called baselines. When there are no loads between control sections, the moment diagram of the equivalent simply supported beam can be drawn only based on the baselines.

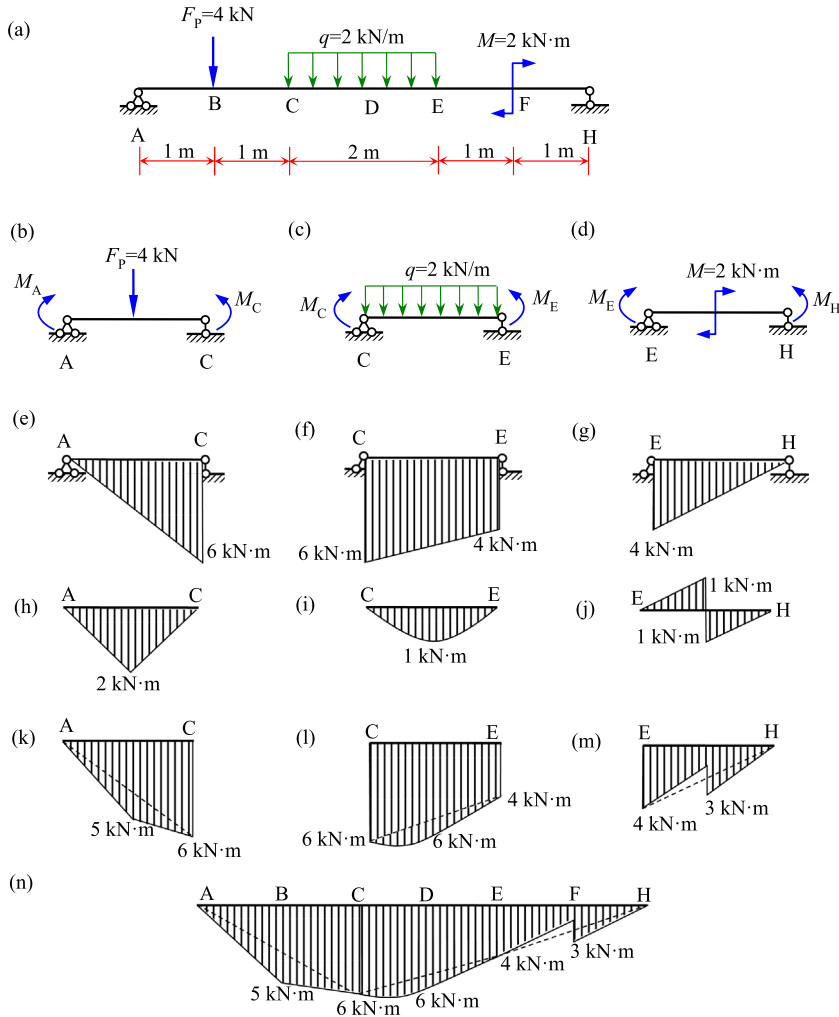


FIG. 3.8 – Force analysis for example 3.1.

- (4) Superposition. When there are loads between the control sections, the baselines should be superimposed with the moment diagram of the equivalent simply supported beam when the loads act alone.

3.2 Multi-Span Statically Determinate Beams

In addition to single-span beams, multi-span beams can be used to cross poor geological areas in engineering applications, as shown in figure 3.9a for concrete multi-span beams of highway bridges. This type of bridge can usually be simplified

as a **multi-span statically determinate beam** to analyze its force condition, and figure 3.9b gives its simplified calculation diagram.

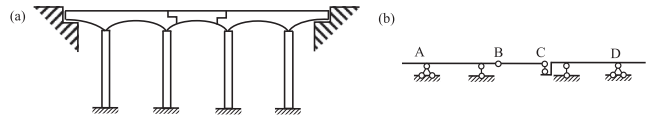


FIG. 3.9 – (a) Multi-span statically determinate beam for highway bridge, (b) the simplified calculation diagram.

From the perspective of geometrical construction, a multi-span statically determinate beam can be divided into basic parts and auxiliary parts. The basic part does not depend on other parts, and it can bear loads independently and maintain geometrical stability. On the contrary, the auxiliary part must rely on the support of other parts to bear the load and maintain geometrical stability. Figure 3.10 demonstrates two construction ways of multi-span statically determinate beams. As shown in figure 3.10a, the left cantilever beam AB is the basic part, and the remaining segments BC and CD are the auxiliary parts of segment AB in turn. Figure 3.10b is composed of cantilever beams AB and CD and a hanging beam BC (such as a wooden purlin of the roof frame), and the cantilever beam AB is the basic part. Although the cantilever beam CD has only two roller supports, it can maintain equilibrium independently under the action of vertical loads. Therefore, the cantilever beam CD is also a basic part under vertical loads, and the hanging beam BC is an auxiliary part.

To clearly show the supporting relation among various parts, the basic part can be drawn at the lower layer, and the auxiliary parts can be drawn at the upper layer. This kind of diagram is called a cascade diagram. Figure 3.11 shows the cascade diagrams of the two multi-span statically determinate beams shown in figure 3.10.

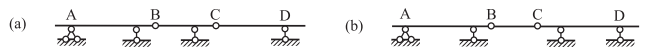


FIG. 3.10 – Two construction ways of multi-span statically determinate beams.

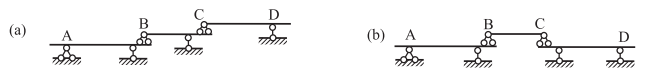


FIG. 3.11 – Cascade diagrams of two multi-span statically determinate beams.

From the perspective of the force analysis, the internal forces of a multi-span statically determinate beam should be analyzed in the order of load transfer (or the order opposite to the geometrical construction): first, calculate the auxiliary part, and then calculate the basic part. Reverse the support reactions of the auxiliary part

to the opposite direction, which are the loads applied to the basic part. In this way, the multi-span statically determinate beam can be disassembled into single-span statically determinate beams, which can be solved one by one, to avoid solving the simultaneous equations. By joining the internal force diagrams of the single-span beams together, the internal force diagrams of the multi-span statically determinate beams are obtained.

Example 3.2. Draw the internal force diagrams of the multi-span statically determinate beam shown in figure 3.12a.

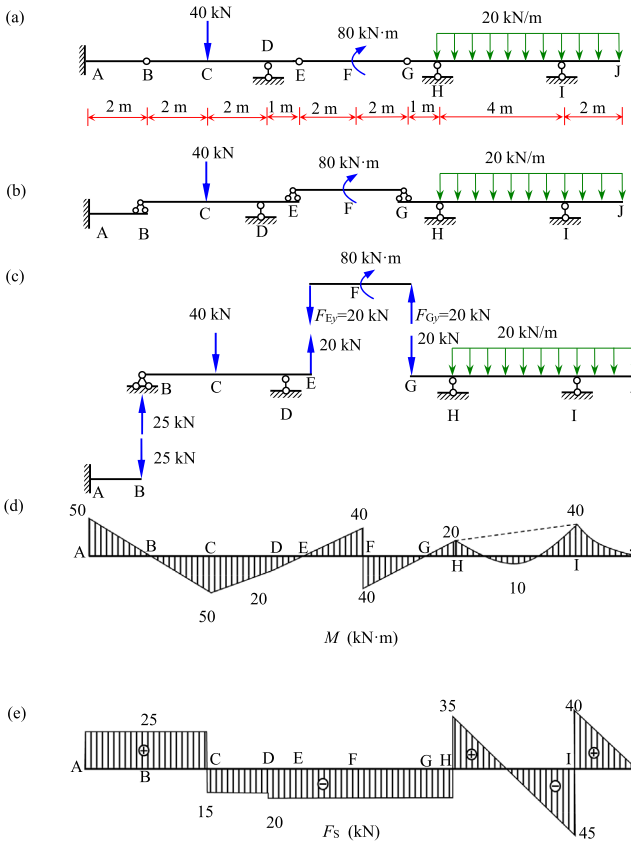


FIG. 3.12 – Multi-span statically determinate beam for example 3.2.

Solution: The geometrical construction order of the multi-span statically determinate beam is analyzed first. Under vertical loads, both beams AB and GJ can maintain geometrical stability, which is the basic parts, while beams BE and EG need to rely on other parts to bear the load, which are the auxiliary parts, and the cascade diagram is shown in figure 3.12b. The calculation should be performed in the order of the cascade diagrams. First, analyze beam EG, and then analyze beams BE and GJ, and finally analyze beam AB.

After the support reactions of beam EG are obtained, the reaction at point E is applied to beam BE in the opposite direction, and the reaction at point G is applied to beam GJ in the opposite direction. In the same way, the reaction at point B of beam BE is reversely applied to beam AB. The magnitude of the load and its transfer path are shown in figure 3.12c. Then, the moment diagrams for each single-span beam can be made with the method of segmental superposition, and then the moment and shear diagrams can be drawn according to the differential relations between the loads and the internal forces or the equilibrium conditions, as shown in figure 3.12d, e.

In general, the solution of the internal forces of a multi-span statically determinate beam firstly needs to solve the interaction forces among single-span beams, and then to calculate the internal force distribution of each single-span beam. In addition, the relations between the moment diagrams and the loads, supports, and joints can also be used, and no or only a part of the constraint forces need to be solved, for quickly drawing the internal force diagrams of the multi-span statically determinate beam, as shown in example 3.3.

Example 3.3. Draw the moment diagram of the multi-span statically determinate beam shown in figure 3.13.

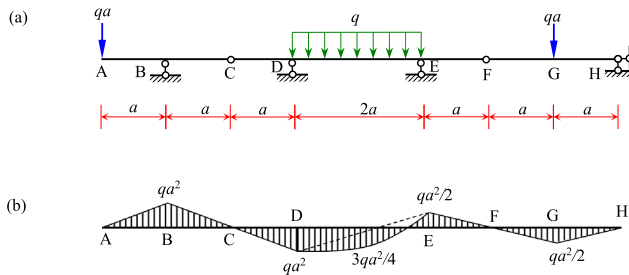


FIG. 3.13 – Multi-span statically determinate beam for example 3.3.

Solution: Firstly, observe the constraints and loads of the multi-span statically determinate beam. Among them, point A is the free end of a cantilever, the moment of which is zero, and the moment of point B can be easily determined as qa^2 , the upper side of which is in tension. Points C, F and H are hinges, and their moments are also zero. Beam FH is equivalent to a simply supported beam, the midpoint G of which is subjected to a concentrated load qa . The moment of G is $qa^2/2$, and its lower side is in tension. So far, the moments of points A, B, C, F, G, and H have been determined. There is no load on the AB and GH segments, so their moment diagrams can be drawn by connecting the moments of points A, B and G, H with straight lines. The segments BD and EG also have no loads, and their moment diagrams should also be straight lines. For segment BD, its moment diagram can be determined simply by connecting the moments of points B and C and extending to point D. According to the geometrical relations, the moment of point D can be

determined as qa^2 , and the lower side is in tension. Similarly, by connecting the moments of points G and F and extending to point E, then the moment of point E can be determined as $qa^2/2$, and the upper side is in tension. Finally, the moments of points D and E are connected, and the moment diagram when segment DE is treated as a simply supported beam subjected to a uniformly distributed load q is superimposed with the method of segmental superposition. Finally, the moment diagrams of segment DE and the entire multi-span statically determinate beam are determined, as shown in figure 3.13b.

Example 3.4. Figure 3.14a illustrates a two-span beam with a uniformly distributed load q . Determine the location of hinge D so that the peak value of the negative moment is equal to that of the positive moment.

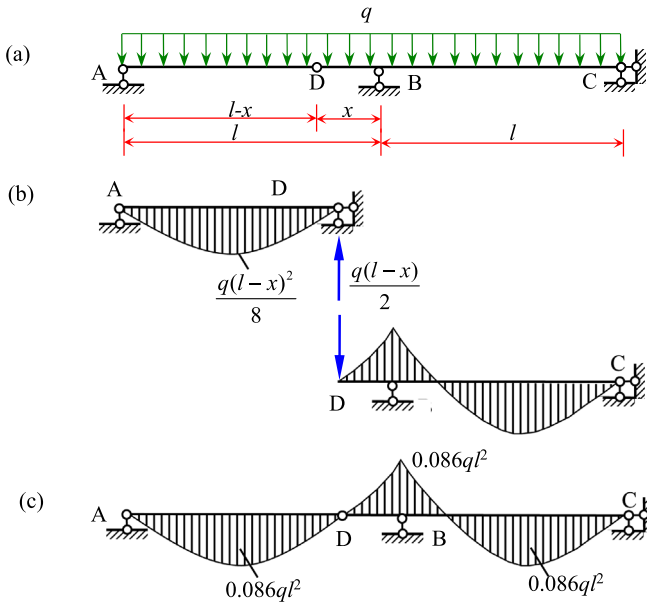


FIG. 3.14 – Two-span beam for example 3.4.

Solution: Let x be the distance between hinge D and support B. In figure 3.14b, first analyze the auxiliary part AD, the support reaction of which is $\frac{q(l-x)}{2}$, and draw the moment diagram, in which the peak value of the moment is $\frac{q(l-x)^2}{8}$.

Then analyze the basic part DC and reverse the direction of the reaction at point D of the auxiliary part to act on the basic part. The peak value of the negative moment is $\frac{q(l-x)x}{2} + \frac{qx^2}{2}$.

Make the positive and negative moment peaks equal to each other, that is,

$$\frac{q(l-x)x}{2} + \frac{qx^2}{2} = \frac{q(l-x)^2}{8}$$

Then

$$x = 0.172l$$

After the location of the hinge is determined, the moment diagram can be drawn as shown in figure 3.14c, where the peak values of both positive and negative moments are equal to $0.086ql^2$.

3.3 Statically Determinate Plane Trusses

3.3.1 Characteristics and Classification of Trusses

When a beam is subjected to loads, the main internal force is the moment. However, the stress distribution on the section of the beam is not uniform, so the material strength cannot be fully utilized. A truss is a lattice system composed of members. When loads only act on the joints, each member only has axial force, and the stresses on the section are evenly distributed, so the material strength can be fully utilized and trusses can be used to cross a large span. Therefore, truss structures are often applied in large-span factory buildings, stadiums, and bridges. Herein, the characteristics and internal force analysis of **statically determinate plane truss** are discussed.

The truss structure is often simplified as an ideal truss during analysis. An ideal truss should meet the following three conditions:

- (1) Joints of the truss are smooth hinges, and each member can slightly rotate around the hinge.
- (2) The axis of each member is straight and passes through the center of the hinge.
- (3) Both the loads and reactions act on the hinge.

However, the truss structure usually deviates from the ideal condition in actual engineering applications. For example, the joints of steel trusses are usually riveted or welded, and the angles among members can hardly change. The axes of truss members are not always straight, and all the axes at the joints do not completely intersect at one point. Moreover, the self-weight and wind load of the members do not act on the joints. However, truss members are generally long and thin, and they are mainly subjected to axial forces when they meet the geometrical construction characteristics of the truss, so the axial forces can often be calculated with the calculation diagram of the ideal truss. In engineering design, member axial forces calculated according to the ideal truss are generally called main internal forces, and additional internal forces due to the difference between the actual situation and ideal condition are called secondary internal forces (mainly the secondary moments). Only the solution to the main internal forces is discussed herein. It should be noted that for modern trusses, the members are usually nearly rigidly connected, although they are assumed pin-connected for simplification analysis with neglecting the very small errors from the second internal forces.

Figure 3.15 displays the simplified calculation diagram of a statically determinate plane truss, and its members can be divided into chord members and web

members according to their locations. The chord members are divided into top and bottom chords, and the web members are divided into vertical and diagonal bars. The horizontal distance between two adjacent joints on the chord member is called the panel. The horizontal distance between the supports is called the span length, and the maximum vertical distance between the joints of the top and bottom chords is called the truss height.

Statically determinate plane trusses can be classified according to different characteristics. According to the characteristics of geometrical construction, they can be divided into three categories:

- (1) Simple truss: a truss originated from the base or a basic hinge-connected triangle, and then constructed by the addition of pin-jointed members, as shown in figure 3.16a, b.
- (2) Compound truss: a truss is composed of several simple trusses according to the basic construction rules of geometrically unchangeable systems, as shown in figure 3.16c.
- (3) Complex truss: other statically determinate trusses that do not fall into the first two categories, as shown in figure 3.16d.

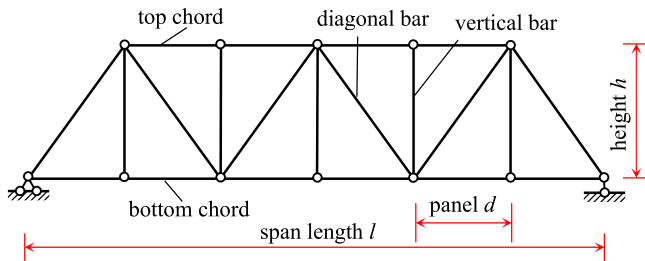


FIG. 3.15 – Simplified calculation diagram of a statically determinate plane truss.

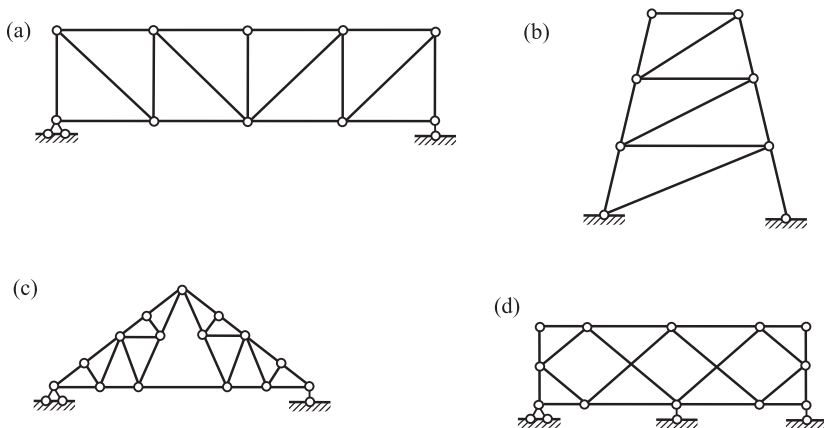


FIG. 3.16 – (a, b) Simple truss, (c) compound truss, and (d) complex truss.

According to the shape, the trusses are also divided into parallel chord truss (figure 3.17a), triangular truss (figure 3.17b), parabolic truss (figure 3.17c), and trapezoidal truss (figure 3.17d).

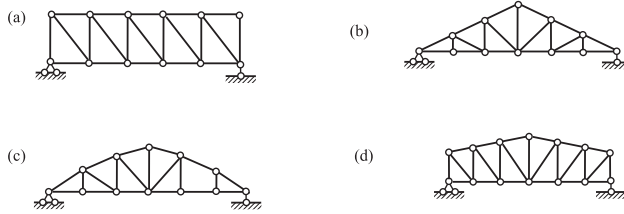


FIG. 3.17 – (a) Parallel chord truss, (b) triangular truss, (c) parabolic truss, and (d) trapezoidal truss.

3.3.2 Method of Joints

The main method of the internal force analysis of statically determinate plane trusses is to intercept a part of the truss as a free body and to solve the unknown member axial forces based on the equilibrium conditions (the tensile force is positive). If the free body contains only one joint, it is called the method of joints. If the free body contains two or more joints, it is called the method of sections.

For statically determinate trusses, the computational degree of freedom is 0. Taking the joints of a truss as parts, let its number be j . Taking the members as constraints (including rods), let its number be b . Then, according to equation (2.7)

$$W = 2j - b = 0$$

That is $2j = b$. All forces acting on any joint of the truss, including loads, support reactions, and axial forces, form a balanced concurrent force system. There are two equilibrium conditions for the force system: $\Sigma F_x = 0$ and $\Sigma F_y = 0$, so each joint has 2 equilibrium equations. Therefore, j joints have $2j$ independent equilibrium equations, and all the unknown forces of the b constraint members can be determined by solving the simultaneous equations.

The method of joints uses a single joint as a free body, which only has two equilibrium equations, and two unknown forces can be determined. Therefore, when the number of unknown forces on the joint does not exceed two, the method of joints is appropriate. Otherwise, the equilibrium equations of multiple joints must be solved simultaneously. The method of joints is suitable for the analysis of simple trusses, because the simple truss starts from a hinge-connected triangle and is constructed by the addition of pin-jointed members in turn, and the last joint will only contain two members. Therefore, the analysis of such bars can firstly solve the support reactions based on the whole equilibrium conditions of the truss, and then begin from the last joint of the geometrical construction as the free body. Following

the reverse order of the geometrical construction analysis, the joints are treated as free bodies in turn to solve the axial forces of all the bars, avoiding the trouble of solving simultaneous equations.

Example 3.5. Figure 3.18a shows the simplified calculation diagram of a simple truss, under the loads shown in the figure. Determine the axial force of each bar.

Solution: (1) Solve the support reactions

$$F_{yA} = F_{yB} = \frac{1}{2} \times (80 \text{ kN} \times 3) = 120 \text{ kN} (\uparrow)$$

(2) The simple truss can be viewed as the combination of the hinge-connected triangle ACD and successively added pin-joined members. The analysis can be performed sequentially from the last joint B in the order opposite to the geometrical construction.

First take joint B as a free body, as shown in figure 3.18b, and the unknown forces are F_{NBH} and F_{NBG} , which are assumed to be tensile forces. The axial force F_{NBG} of the diagonal bar is replaced by its component forces F_{xBG} and F_{yBG} . From $\Sigma F_y = 0$, $120 \text{ kN} + F_{yBG} = 0$, so $F_{yBG} = -120 \text{ kN}$. Using the proportional relation, $F_{xBG} = -120 \text{ kN} \times 3/4 = -90 \text{ kN}$. Finally, from $\Sigma F_x = 0$, $F_{NBH} + F_{xBG} = 0$, and $F_{NBH} = 90 \text{ kN}$ (tension).

(3) Analyze joint H

The free body diagram of joint H is shown in figure 3.18c, where the known forces are drawn in the actual directions, and the unknown forces F_{NHF} and F_{NHG} are assumed to be tensile forces. From $\Sigma F_x = 0$, $F_{NHF} - 90 \text{ kN} = 0$ and $F_{NHF} = 90 \text{ kN}$ (tension). From $\Sigma F_y = 0$, $80 \text{ kN} - F_{NHG} = 0$ and $F_{NHG} = 80 \text{ kN}$ (tension).

(4) Analyze joints G and E in turn

The free body diagram is shown in figure 3.18d. The unknown forces are F_{NGE} and F_{NGF} , and the axial force F_{NGF} of the diagonal bar is replaced by the component forces F_{xGF} and F_{yGF} . From $\Sigma F_y = 0$, $F_{yGF} + 80 \text{ kN} - 120 \text{ kN} = 0$, so $F_{yGF} = 40 \text{ kN}$. From the geometric proportional relation between F_{xGF} and F_{yGF} , $F_{xGF} = 40 \text{ kN} \times 3/4 = 30 \text{ kN}$. Finally, from $\Sigma F_x = 0$, $F_{NGE} + 30 \text{ kN} + 90 \text{ kN} = 0$, and finally $F_{NGE} = -120 \text{ kN}$ (compression).

Then Analyze joint E, from $\Sigma F_y = 0$, $F_{NEF} = -80 \text{ kN}$ (compression).

(5) Using symmetry

Since both the truss and loads are symmetrical, the internal forces of the truss should also be symmetrical. Therefore, the axial forces of other bars can be directly obtained from the calculated member axial forces according to the symmetry. The axial forces of the entire truss are exhibited in figure 3.18e.

The above is the general process of the method of joints. In addition, when applying the method of joints, special properties of trusses or joints can also be used to speed up the solving process of the internal forces.

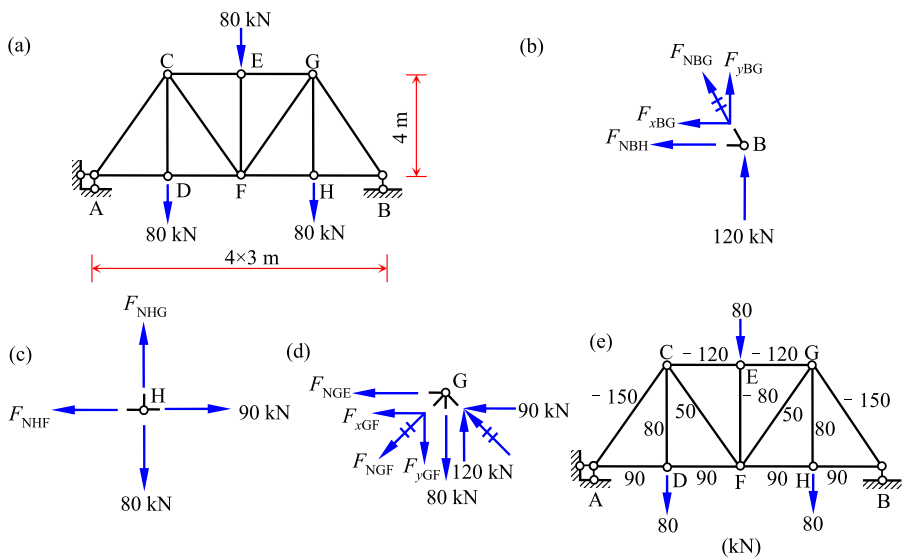


FIG. 3.18 – Simple truss for example 3.5.

The first is to use the nature of the joint single-bar to directly determine some members with zero axial force in the truss, which is the so-called zero-force bar, or to directly determine the relations between the axial forces of certain members. A member is called the joint single-bar, the internal forces of which can be determined by one equilibrium equation when the joint is treated as a free body. When the joint has no load, the axial force of the single bar is zero. Based on this, the zero-force bars in the truss shown in figure 3.19 can be quickly determined (the bars labeled with two short lines perpendicular to the member axis are zero-force bars), simplifying the calculation of the axial forces.

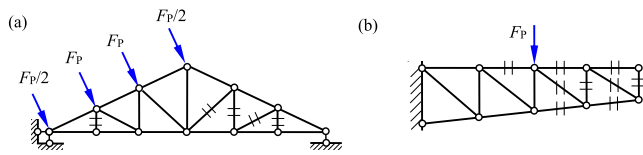


FIG. 3.19 – Quick determination of the zero-force bars.

The second is to use symmetry. The symmetry of a truss structure can be used to simplify the calculation of the axial forces. If symmetrical structures are subjected to symmetrical or anti-symmetrical loads, their internal forces and deformations (also called responses) must also be symmetrical or anti-symmetrical. The using of symmetry not only will directly reduce the calculation of the structural analysis but also can determine the force states of some special members, which also serves the

purpose of simplifying the calculation. As shown in figure 3.20a, the truss is a symmetrical structure under the action of symmetrical loads, and then its axial force distribution is also symmetrical with regard to the symmetry axis. Then, combined with the vertical equilibrium condition at point D, members CD and ED can be directly determined as zero-force bars. In figure 3.20b, the symmetrical structure is subjected to anti-symmetrical load, and its axial forces must be anti-symmetrically distributed with regard to the symmetry axis, so member CE can be directly determined as a zero-force bar.

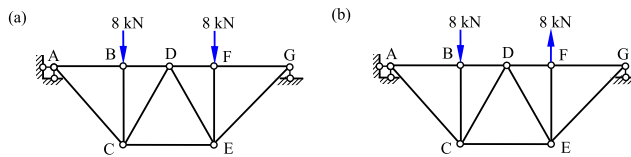


FIG. 3.20 – Symmetrical structures subjected to (a) symmetrical and (b) anti-symmetrical loads.

3.3.3 Method of Sections

As mentioned above, the method of joints uses joints as the free body, with which only 2 unknown forces can be determined each time, and a truss needs to be solved according to the construction order. When a joint contains 3 or more unknown forces or only the axial forces of some members of the truss need to be solved, the method of joints is not suitable. Herein, the method of sections can be used. The method of sections uses a section to cut the members and to separate one part from other parts to form a free body, and the free body includes two or more joints. Herein, all the forces acting on the free body form a plane general force system, and three independent equilibrium equations can be established. In this way, if there are no more than three unknown forces on the free body, these three equilibrium equations can be used to solve the unknown forces.

The method of sections is suitable for simple trusses and compound trusses. Firstly, select an appropriate section, which will truncate no more than 3 members, so that the unknown axial forces of the free body can be determined. Secondly, when the equilibrium equations are established, try to make each equation contains only 1 unknown force so that the solution of simultaneous equations can be avoided. Finally, the property of a sectional single-bar can be utilized. Regardless of the number of unknown axial forces on the section which is taken out by using the method of sections, if the axial force of a member can be determined by only one equilibrium equation, the member is called a sectional single-bar. Therefore, in some cases, more than three members can be truncated with one section. For example, among all the truncated members, other members intersect at one point or are parallel to each other except for one member, then the member is a sectional single-bar, and its axial force can still be determined by solving one equilibrium equation. As shown in figure 3.21, member a is a single bar of the section m-m.

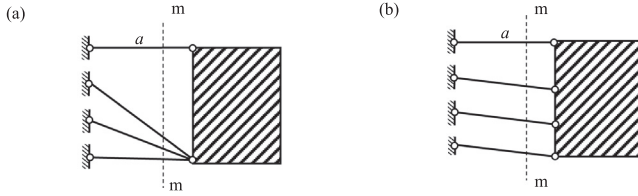


FIG. 3.21 – Sectional single-bars.

Example 3.6. Determine the axial forces of bars 1, 2, 3, and 4 of the truss under the loads shown in figure 3.22.

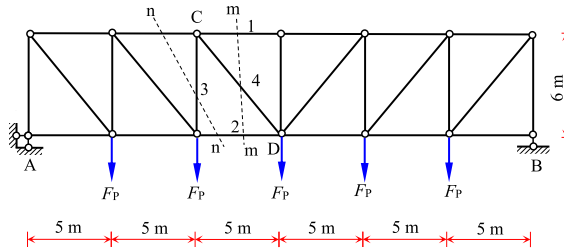


FIG. 3.22 – Plane truss for example 3.6.

Solution: The first is to solve the support reactions, and it is easy to get the two vertical support reactions according to the symmetry, which both are $2.5F_P$ ($\uparrow$).

Firstly, cut bars 1, 2, and 4, make the m-m section as shown in the figure, and divide the truss into two parts. Take the left part as a free body, the external loads of which includes the support reactions, the loads, and the unknown axial forces of bars 1, 2, and 4 (each one is a single member of the section). The free body can give 3 equilibrium equations, so 3 unknown forces can be solved. However, it is necessary to avoid solving simultaneous equations when trying to set equilibrium equations, so proper equilibrium equations should be chosen and solved in order. Firstly, the intersection point D of bars 2 and 4 can be the center to set the moment equilibrium equation. In this way, the moments of the axial forces of bars 2 and 4 for point D are zero, and the equilibrium equation has only one unknown axial force of bar 1. Then

$$\sum M_D = 0, \quad F_{yA} \times 15 \text{ m} - F_P \times (10 \text{ m} + 5 \text{ m}) + F_{N1} \times 6 \text{ m} = 0$$

$F_{yA} = 2.5 F_P$, and then $F_{N1} = -3.75F_P$ (compression).

Similarly, the intersection point C of bars 1 and 4 can be the center of the moment equilibrium equation, and the axial force of bar 2 can be obtained:

$$\sum M_C = 0, \quad F_{yA} \times 10 \text{ m} - F_P \times 5 \text{ m} - F_{N2} \times 6 \text{ m} = 0$$

Solve the equation, $F_{N2} = 3.33F_P$ (tension).

After solving F_{N1} and F_{N2} , the vertical component F_{yN4} of F_{N4} can be determined according to the vertical projection equilibrium equation of the free body

$$\sum F_y = 0, \quad F_{yN4} + 2F_P - 2.5F_P = 0$$

Therefore, $F_{yN4} = 0.5F_P$. Based on the geometric relations, $F_{N4} = 0.65F_P$.

To solve the axial force of bar 3, another section n-n should be chosen as shown in the figure. The left part is treated as the free body, and the vertical projection equilibrium equation is set to get $F_{N3} = -0.5F_P$.

3.3.4 Combined Application of the Method of Joints and the Method of Sections

For the internal force calculation of some trusses, sometimes it is difficult using the method of joints or the method of sections alone. They can be combined to quickly calculate the axial forces of the specific members, which is named the combined method.

Figure 3.23a illustrates a simple truss. Calculate the axial forces of bars 1 and 2.

When using the method of sections, the section n-n can be taken and the left half part is treated as the free body, and the equation obtained from $\sum F_y = 0$ includes two unknown quantities F_{y1} and F_{y2} , so another equation needs to be added. This supplementary equation can be obtained using the method of joints. Herein, joint D is taken as the free body, as shown in figure 3.23b, and a second equation including F_{y1} and F_{y2} can be established based on the joint equilibrium equation. If the two equations are combined, the axial forces of bars 1 and 2 can be solved.

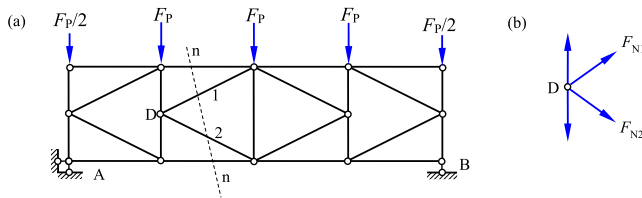


FIG. 3.23 – Internal force calculation of a simple truss *via* combined use of method of joints and method of sections.

Example 3.7. Calculate the axial forces of bars 1, 2, and 3 of the truss shown in figure 3.24a.

Solution: Calculate the support reactions first.

This is a compound truss, which is not easy to solve with the method of joints alone.

Section m-m is made firstly to calculate F_{N4} . The right part of section m-m is treated as the free body (the free body figure is omitted), and G is treated as the moment center. Then

$$\sum M_G = 0, \quad F_{N4} \times 4 \text{ m} - 6 \text{ kN} \times 8 \text{ m} = 0$$

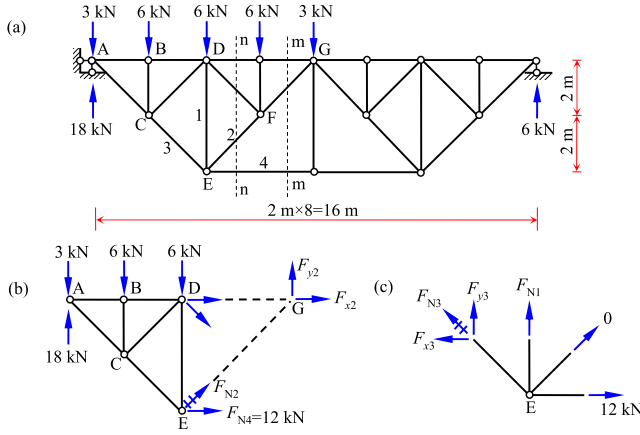


FIG. 3.24 – Compound truss for example 3.7.

Therefore

$$F_{N4} = 12 \text{ kN (tension)}$$

Make another section n-n, and take the left part as a free body (figure 3.24b). Take point D as the moment center and decompose F_{N2} into F_{x2} and F_{y2} at point G.

$$\sum M_D = 0, \quad 18 \text{ kN} \times 4 \text{ m} - 3 \text{ kN} \times 4 \text{ m} - 6 \text{ kN} \times 2 \text{ m} - 12 \text{ kN} \times 4 \text{ m} - F_{y2} \times 4 \text{ m} = 0$$

Therefore

$$F_{y2} = 0, \quad F_{x2} = 0, \quad F_{N2} = 0$$

Then take joint E as a free body (figure 3.24c), and use the projection equation to determine F_{N1} and F_{N3} .

$$\sum F_x = 0, \quad 12 \text{ kN} - F_{x3} = 0$$

Therefore

$$F_{x3} = 12 \text{ kN}$$

Then

$$F_{y3} = 12 \text{ kN}, \quad F_{N3} = 12 \text{ kN} \times \sqrt{2} = 16.97 \text{ kN (tension)}$$

$$\sum F_y = 0, \quad F_{N1} + F_{y3} = 0$$

Then

$$F_{N1} = -F_{y3} = -12 \text{ kN (compression)}$$

3.4 Statically Determinate Plane Frames

3.4.1 Characteristics of Frames

A frame is a member structure composed of beams and columns, the whole or part of which are connected by rigid joints. When the member axes, loads, and support reactions of the frame are in the same plane and there are no redundant constraints, it is called a **statically determinate plane frame**. From the perspective of deformation, there are no relative rotations of all the member ends at the rigid joints of the frame, so the angles among the members remain unchanged. From the perspective of force, rigid joints can transmit the axial force, shear, and moment, and the moment is the main internal force in the frame. A frame has good integrity, great stiffness, and large effective usable space, and is widely used in engineering applications.

Common types of statically determinate plane frames are cantilever frames (figure 3.25a), simply supported frames (figure 3.25b), three-hinged frames (figure 3.25c), and master-slave frames (figure 3.25d).

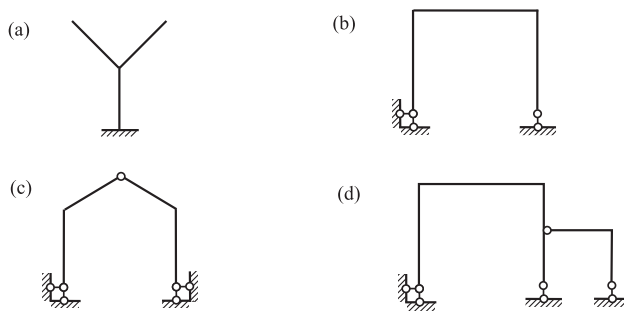


FIG. 3.25 – (a) Cantilever frame, (b) simply supported frame, (c) three-hinged frame, and (d) master-slave frame.

3.4.2 Calculation of Support Reactions

In the analysis of statically determinate plane frames, usually, the support reactions are first determined, then the internal forces of the key sections of each member are obtained, and finally, the internal force diagram of the frame is drawn. In the

calculation of the support reactions of the frame, independent equations should be established as much as possible.

For the simply supported frame shown in figure 3.26, the vertical projection equilibrium equation $\sum F_y = 0$ can be established first to obtain the support reactions, and then F_{RC} can be obtained. Then calculate the moments to the intersection point O of F_{RC} and F_{RB} , establish the moment equilibrium equation $\sum M_O = 0$, and solve F_{RA} . Finally, a horizontal equilibrium equation $\sum F_x = 0$ is established, and F_{RB} can be calculated.

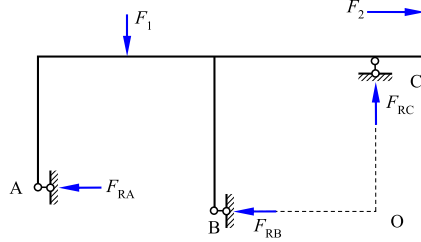


FIG. 3.26 – Calculation of support reactions of simply supported frame.

As shown in figure 3.27a, the three-hinged frame has four support reactions. Then, three whole equilibrium equations and a local equilibrium condition, *i.e.*, the moment at hinge C is zero, are used to determine the 4 support reactions.

Separate the three-hinged frame from the base to form a free body, and establish the moment equilibrium equation at point B:

$$\sum M_B = 0, \quad F_{yA}l + qf \cdot \frac{f}{2} = 0, \quad F_{yA} = -\frac{qf^2}{2l} (\downarrow)$$

Second, construct a vertical projection equilibrium equation for the free body,

$$\sum F_y = 0, \quad F_{yA} + F_{yB} = 0, \quad F_{yB} = \frac{qf^2}{2l} (\uparrow)$$

Again, establish the horizontal projection equilibrium equation for the free body, and the relation between F_{xA} and F_{xB} can be obtained:

$$\sum F_x = 0, \quad F_{xA} + qf - F_{xB} = 0, \quad F_{xA} = F_{xB} - qf$$

Finally, F_{xA} and F_{xB} can be determined using the equilibrium equation, *i.e.*, the moment at hinge C is zero. If the right half of the rigid frame is treated as a free body, then

$$\sum M_C = 0, \quad F_{xB}f - F_{yB} \frac{l}{2} = 0, \quad F_{xB} = \frac{qf}{4} (\leftarrow)$$

Therefore

$$F_{xA} = -\frac{3}{4}qf (\leftarrow)$$

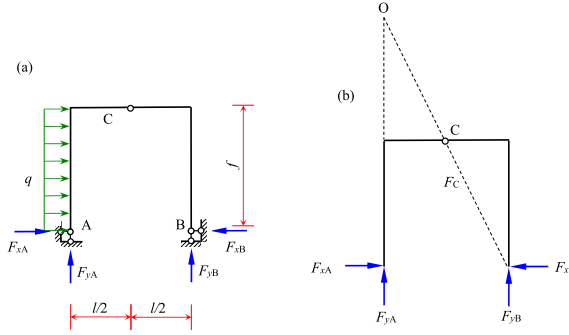


FIG. 3.27 – Calculation of support reactions of three-hinged frame.

It can be seen from the above solution process that generally the support reactions of the three-hinged frame are coupled in pairs and need to be calculated by solving the simultaneous equations. Therefore, establishing independent static equilibrium equations of the support reactions can reduce the computational effort. In the frame shown in figure 3.27a, the right half of the frame has no loads and the moment at point C is zero, so the action line of the reaction at support B must pass through point C. Line BC and the action line of F_{yA} intersect at point O, as shown in figure 3.27b. Then, taking point O as the center to establish the moment equilibrium equations, F_{xA} can be determined and used as the breakthrough for solving other support reactions:

$$\sum M_O = 0, \quad F_{xA} \cdot 2f + qf \cdot \frac{3}{2}f = 0, \quad F_{xA} = -\frac{3}{4}qf (\leftarrow)$$

Figure 3.28a shows a master-slave frame with four support reactions. According to the geometrical construction analysis, the right part of point C is the basic part, and the left part is the auxiliary part. The force analysis should be performed in the reverse order of the geometrical construction, from the auxiliary part to the basic part. The left part of point C is firstly treated as a free body, as shown in figure 3.28b. Taking point C as the center of the moment equilibrium equation, F_{xD} can be obtained:

$$\sum M_C = 0, \quad F_{xD} \cdot a + P \cdot \frac{a}{2} + qa \cdot \frac{1}{2}a = 0, \quad F_{xD} = -qa$$

Then, according to the horizontal projection equilibrium equation of the free body

$$\sum F_x = 0, \quad F_{xD} + P - F_{xC} = 0, \quad F_{xC} = 0$$

Again, according to the vertical projection equilibrium condition of the free body, it can be determined:

$$\sum F_y = 0, \quad F_{yC} - qa = 0, \quad F_{yC} = qa$$

After obtaining the support reactions of the adjunctive part and the constraint forces between the adjunctive part and the basic part, the constraint forces can be reversely acted on the basic part, and then the support reactions of the basic part can be determined, as shown in figure 3.28c. Cut the basic part from the base as a free body, and then the horizontal projection equilibrium equation can be given to solve F_{xA} :

$$\sum F_x = 0, \quad F_{xA} - Q - F_{xC} = 0, \quad F_{xA} = qa$$

Then, take point A as the center to establish the moment equilibrium equation:

$$\sum M_A = 0, \quad F_{yB} \cdot a + F_{xC} \cdot \frac{5}{4}a + Q \cdot \frac{5}{8}a - F_{yC} \cdot \frac{3}{2}a - q \cdot \frac{3}{2}a \cdot \frac{3}{4}a = 0, \quad F_{yB} = 2qa$$

Finally, according to the vertical projection equilibrium equation of the basic part, F_{yA} is obtained:

$$\sum F_y = 0, \quad F_{yA} + F_{yB} - q \cdot \frac{3}{2}a - F_{yC} = 0, \quad F_{yA} = \frac{1}{2}qa$$

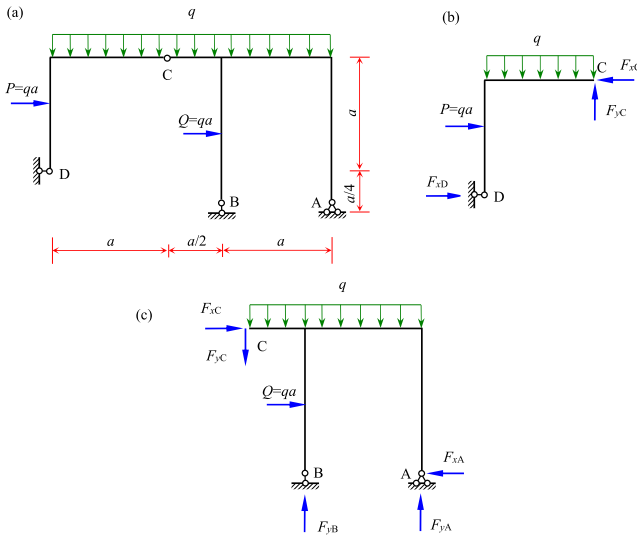


FIG. 3.28 – Calculation of support reactions of master-slave frame.

3.4.3 Internal Force Analysis and Drawing Internal Force Diagram of Frames

After determining the support reactions of the frame, the internal force analysis and drawing an internal force diagram of the frame can be performed, and the steps are usually presented as follows.

- (1) Segmentation: Divide the frame into several segments according to the joints and load discontinuity points.
- (2) Shape determination: Determine the shape of the internal force diagram in terms of the loads in each segment.
- (3) Evaluation: The internal forces of each key section are evaluated with the method of sections or the internal force calculation formulas.
- (4) Drawing: According to the internal forces of key sections and the lateral loads, combined with the method of segmental superposition, draw the internal force diagrams of the frame.

Example 3.8. Solve the support reactions of the simply supported frame shown in figure 3.29a and draw the moment, shear, and axial force diagrams.

Solution: (1) Determine the support reactions of the simply supported frame.

Use the three whole equilibrium conditions to obtain the support reactions: $F_{xA} = 80 \text{ kN}$, $F_{yA} = 20 \text{ kN}$, $F_{yD} = 60 \text{ kN}$.

- (2) Solve the internal forces of member ends and draw the internal force diagrams of each member.

The frame is divided into two parts: members AB and BD, and then they are solved separately, in which the key problem is to determine the internal forces of member ends at joint B. Member AB is taken as a free body, as shown in figure 3.29b.

Establish the equilibrium equations of the free body to determine the internal forces at end B of member AB:

$$\sum F_x = 0, \quad F_{SBA} + 20 \times 4 - F_{xA} = 0, \quad F_{SBA} = 0$$

$$\sum F_y = 0, \quad F_{NBA} - F_{yA} = 0, \quad F_{NBA} = 20 \text{ kN}$$

$$\sum M_B = 0, \quad M_{BA} + 20 \times 4 \times 2 - F_{xA} \times 4 = 0, \quad M_{BA} = 160 \text{ kN} \cdot \text{m}$$

The method of segmental superposition is used to draw the moment diagram of AB shown in figure 3.29c.

Then, using joint B as a free body, as shown in figure 3.29d, the internal forces at end B of member BD can be determined according to the joint equilibrium equations.

$$F_{SBD} = -F_{NBA} = -20 \text{ kN}, \quad F_{NBD} = F_{SBA} = 0 \text{ kN}, \quad M_{BD} = M_{BA} = 160 \text{ kN} \cdot \text{m}$$

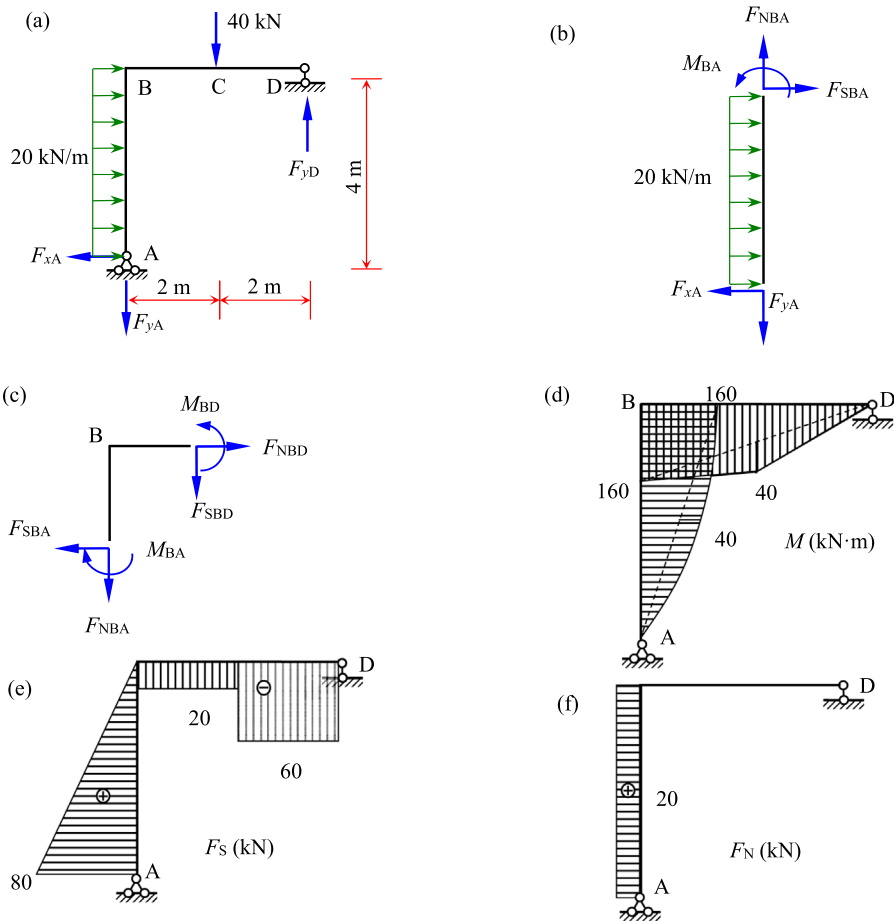


FIG. 3.29 – Simply supported frame for example 3.8.

According to the internal forces at the end of member BD, the method of segmental superposition can be used to determine the internal forces at each section of member BD, and then draw the moment diagram of the entire simply supported frame, as shown in figure 3.29d. The shear and axial force diagrams of the frame can also be obtained by joining the shear and axial force diagrams of each member, as shown in figure 3.29e, f.

Example 3.9. Draw the moment diagram of the cantilever frame shown in figure 3.30a.

Solution: For the cantilever frame, the support reactions can be calculated first, or the moment diagram of the member can be drawn directly from the cantilever end, which is used in this example. According to the geometrical construction and load

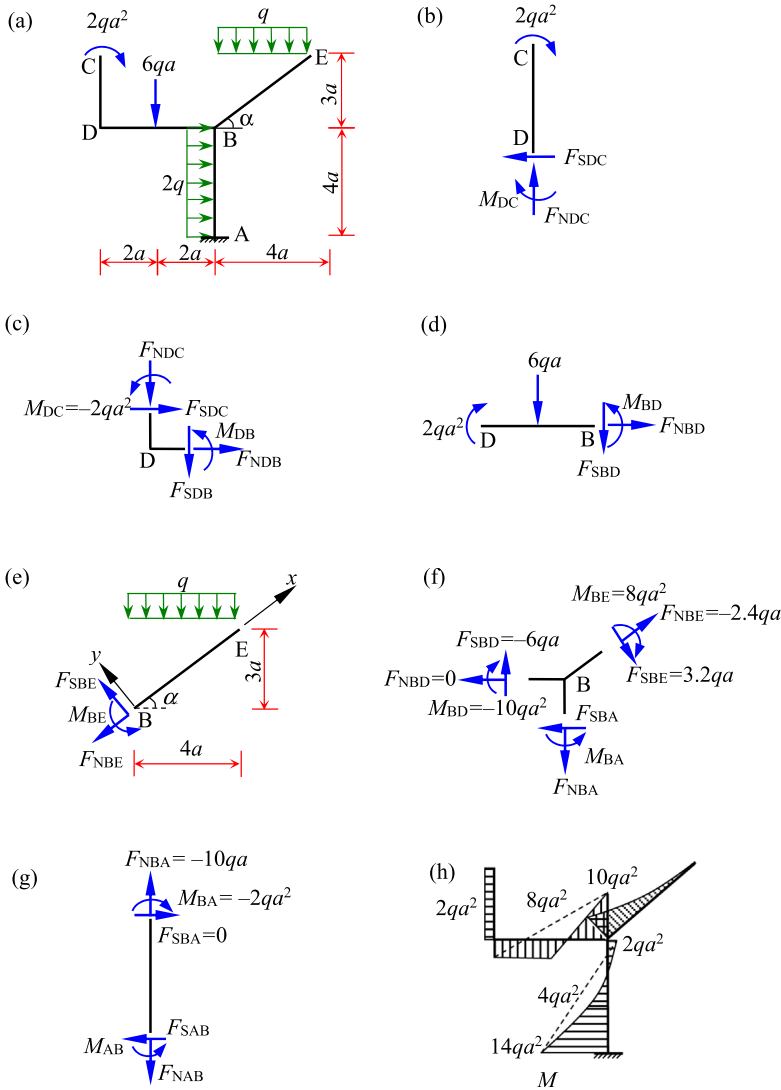


FIG. 3.30 – Cantilever frame for example 3.9.

distribution, the frame can be divided into four members: CD, BE, DB, and BA, and the calculation can be performed from outside to inside.

- (1) First analyze member CD. With member CD as a free body, as shown in figure 3.30b, its equilibrium equation is used to determine the internal forces at end D:

$$F_{SDC} = 0, \quad F_{NDC} = 0, \quad M_{DC} + 2qa^2 = 0, \quad M_{DC} = -2qa^2$$

- (2) Analyze joint D to determine the internal forces at end D of member BD, as shown in figure 3.30c:

$$F_{SDB} = 0, \quad F_{NDB} = 0, \quad M_{DB} = -M_{DC} = 2qa^2$$

- (3) Analyze member DB to determine the internal forces at end B of member DB, as shown in figure 3.30d:

$$\begin{aligned} F_{NBD} &= 0 \\ F_{SBD} + 6qa &= 0, \quad F_{SBD} = -6qa \\ \sum M_B &= 0, \quad M_{BD} - M_{DB} + 6qa \cdot 2a = 0, \quad M_{BD} = -10qa^2 \end{aligned}$$

- (4) Analyze the inclined member BE in figure 3.30e, and determine the internal forces at end B by the equilibrium equations:

$$\begin{aligned} \sum F_x &= 0, \quad F_{NBE} + q \cdot 4a \cdot \sin \alpha = 0, \quad F_{NBE} = -4qa \cdot 0.6 = -2.4qa \\ \sum F_y &= 0, \quad F_{SBE} - q \cdot 4a \cdot \cos \alpha = 0, \quad F_{SBE} = 4qa \cdot 0.8 = 3.2qa \end{aligned}$$

$$\sum M_B = 0, \quad M_{BE} - q \cdot 4a \cdot 2a = 0, \quad M_{BE} = 8qa^2$$

- (5) Analyze joint B in figure 3.30f, and establish the equilibrium equations to determine the internal forces at end B of member AB:

$$\begin{aligned} \sum F_x &= 0, \quad F_{SBA} + F_{NBD} - F_{SBE} \cdot \sin \alpha - F_{NBE} \cdot \cos \alpha = 0, \quad F_{SBA} = 0 \\ \sum F_y &= 0, \quad F_{NBA} - F_{SBD} + F_{SBE} \cdot \cos \alpha - F_{NBE} \cdot \sin \alpha = 0, \quad F_{NBA} = -10qa \\ \sum M_B &= 0, \quad M_{BA} - M_{BD} - M_{BE} = 0, \quad M_{BA} = -2qa^2 \end{aligned}$$

- (6) Finally, analyze member AB in figure 3.30g, to find the internal forces at end A of member AB:

$$\begin{aligned} F_{NAB} &= F_{NBA} = -10qa \\ F_{SAB} - F_{SBA} - 2q \cdot 4a &= 0, \quad F_{SAB} = 8qa \\ M_{AB} - M_{BA} - 2q \cdot 4a \cdot 2a &= 0, \quad M_{AB} = 14qa^2 \end{aligned}$$

So far, the internal force analysis of all the member ends of the frame has been completed. Using the method of segmental superposition, the internal force diagrams of each member can be drawn and finally connected to form the internal force diagram of the frame. The moment diagram is shown in figure 3.30h, and the shear and axial force diagrams can also be drawn similarly.

Example 3.10. Draw the moment diagram of the three-hinged frame shown in figure 3.31a.

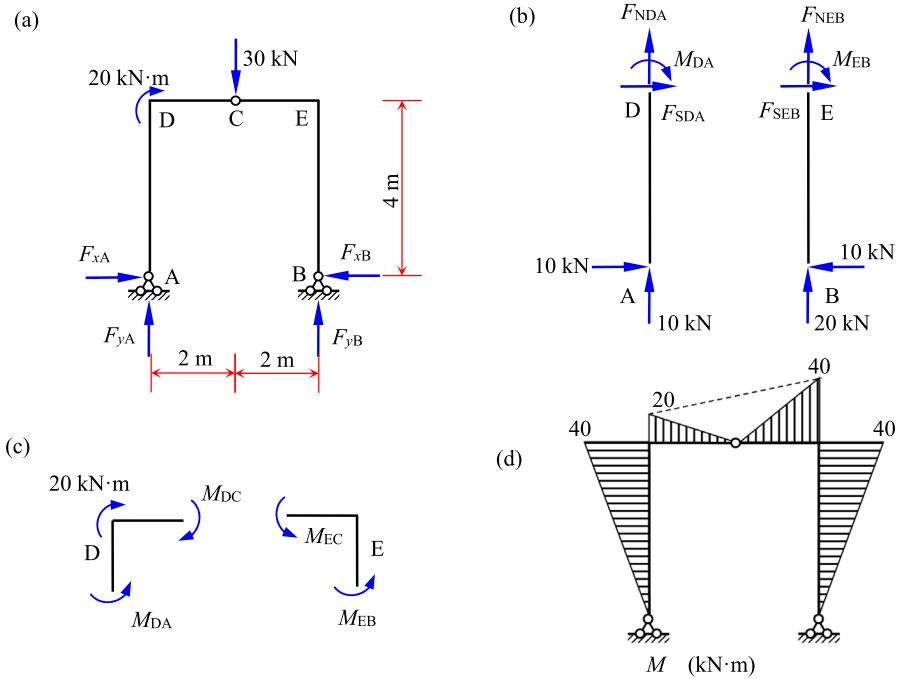


FIG. 3.31 – Three-hinged frame for example 3.10.

Solution: (1) Solve the support reactions

$$\begin{aligned}\sum M_B &= 0, \quad F_{yA} \cdot 4 + 20 - 30 \cdot 2 = 0, \quad F_{yA} = 10 \text{ kN} \\ \sum F_y &= 0, \quad F_{yB} = 30 - F_{yA} = 20 \text{ kN} \\ \sum M_C &= 0, \quad F_{yB} \cdot 2 - F_{xB} \cdot 4 = 0, \quad F_{xB} = 10 \text{ kN} \\ \sum F_x &= 0, \quad F_{xA} = F_{xB} = 10 \text{ kN}\end{aligned}$$

(2) Draw the moment diagram of the frame. The key problem is to determine the moments of member ends at joints D and E, so members AD and BE are analyzed first, as shown in figure 3.31b. For member AD,

$$\sum M_D = 0, \quad M_{DA} = F_{xA} \cdot 4 = 40 \text{ kN}\cdot\text{m}$$

For member BE,

$$\sum M_E = 0, \quad M_{EB} = -F_{xB} \cdot 4 = -40 \text{ kN}\cdot\text{m}$$

Then, according to the joint equilibrium conditions, as shown in figure 3.31c, M_{DC} and M_{EC} can be determined:

$$M_{DC} = M_{DA} - 20 = 20 \text{ kN}\cdot\text{m}$$

$$M_{EC} = -M_{EB} = 40 \text{ kN}\cdot\text{m}$$

According to the property of the hinge, the moment at point C $M_C = 0$. According to the method of segmental superposition, the moment diagram of each segment can be directly drawn, and finally, the moment diagram of this three-hinged frame can be combined, as shown in figure 3.31d.

Example 3.11. Draw the moment diagram of the master-slave frame shown in figure 3.32a.

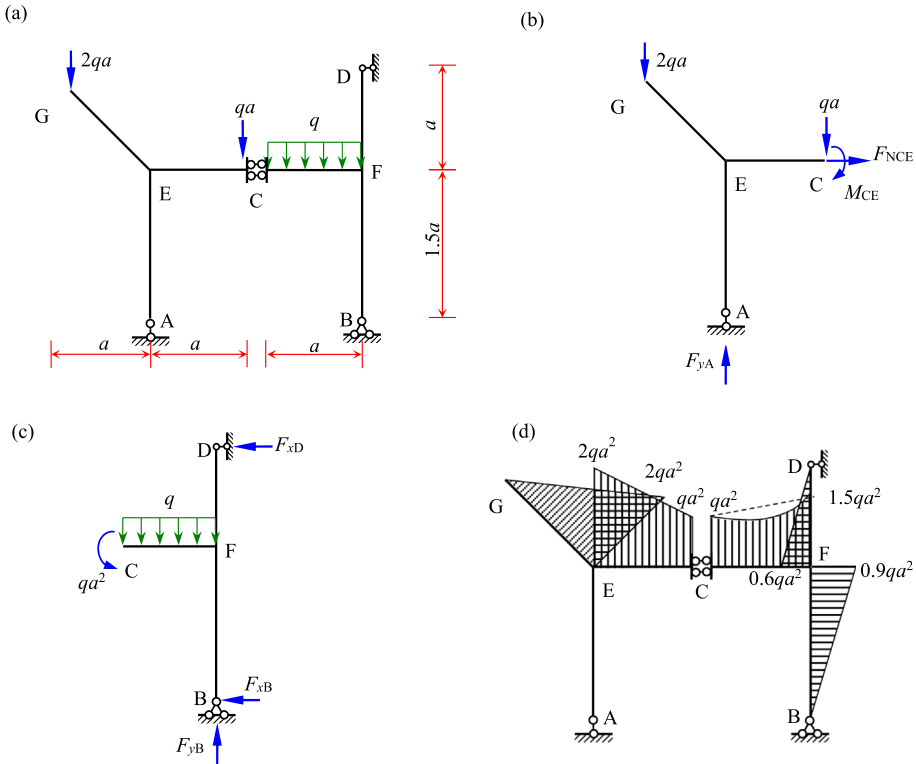


FIG. 3.32 – Master-slave frame for example 3.11.

Solution: For the master-slave frame, the adjunctive part should be calculated first, then the basic part. As shown in figure 3.32a, the frame AGE is the adjunctive part and acts on the basic part CDB through point C. Therefore, AGE should be

analyzed firstly as a free body in figure 3.32b. After determining the internal forces at point C of the adjunctive part, the internal forces can be used to react on the basic part, and then the basic part can be analyzed as a free body in figure 3.32c. After that, the basic part can be split into single members to determine the internal forces of the key joint F. Finally, the bending moment diagram of the entire structure is determined and drawn according to the method of segmental superposition, as shown in figure 3.32d.

After completing the moment diagram, the indirect method can also be used to determine the shears and axial forces of the frame, the basic idea of which is to take a member as a free body, establish moment equilibrium equations, calculate the shears based on the moments of member ends, and draw the shear diagram. Finally, taking the joint as a free body, the axial force is obtained from the shears of the member ends using the projection equilibrium, and then the axial force diagram can be drawn.

Example 3.12. Draw the internal force diagrams of the portal frame under a uniformly distributed load in figure 3.33a.

Solution: (1) Solve the support reactions of frame

$$\begin{aligned}\sum M_A &= 0, \quad 6 \text{ kN} \times 3 \text{ m} - F_{yB} \times 12 \text{ m} = 0, \quad F_{yB} = 1.5 \text{ kN} (\uparrow) \\ \sum M_B &= 0, \quad 6 \text{ kN} \times 9 \text{ m} - F_{yA} \times 12 \text{ m} = 0, \quad F_{yA} = 4.5 \text{ kN} (\uparrow) \\ \sum F_x &= 0, \quad F_{xA} - F_{xB} = 0, \quad F_{xA} = F_{xB}\end{aligned}$$

$$\sum M_C = 0 \quad (\text{Consider the right part of hinge C}),$$

$$\begin{aligned}F_{xB} \times 6.5 \text{ m} - F_{yB} \times 6 \text{ m} &= 0, \quad F_{xB} = 1.385 \text{ kN} (\leftarrow) \\ F_{xA} &= 1.385 \text{ kN} (\rightarrow)\end{aligned}$$

(2) Draw the moment diagram M

Firstly, calculate the moments of member ends, draw them on the tensile side of the member axes and connect them with straight lines, and then superimpose the moment diagram of the simply supported beam.

Take member DC as an example,

$$\begin{aligned}M_{DC} &= -1.385 \text{ kN} \times 4.5 \text{ m} = -6.23 \text{ kN} \cdot \text{m} \\ M_{CD} &= 0\end{aligned}$$

The moment of the center point of member DC is

$$\frac{1}{2} \times (-6.23 \text{ kN} \cdot \text{m}) + \frac{1}{8} \times 1 \text{ kN/m} \times (6 \text{ m})^2 = 1.38 \text{ kN} \cdot \text{m}$$

The moment diagram M is shown in figure 3.33b.

(3) Draw the shear diagram F_s

The shear of a member end can be solved with direct and indirect methods.

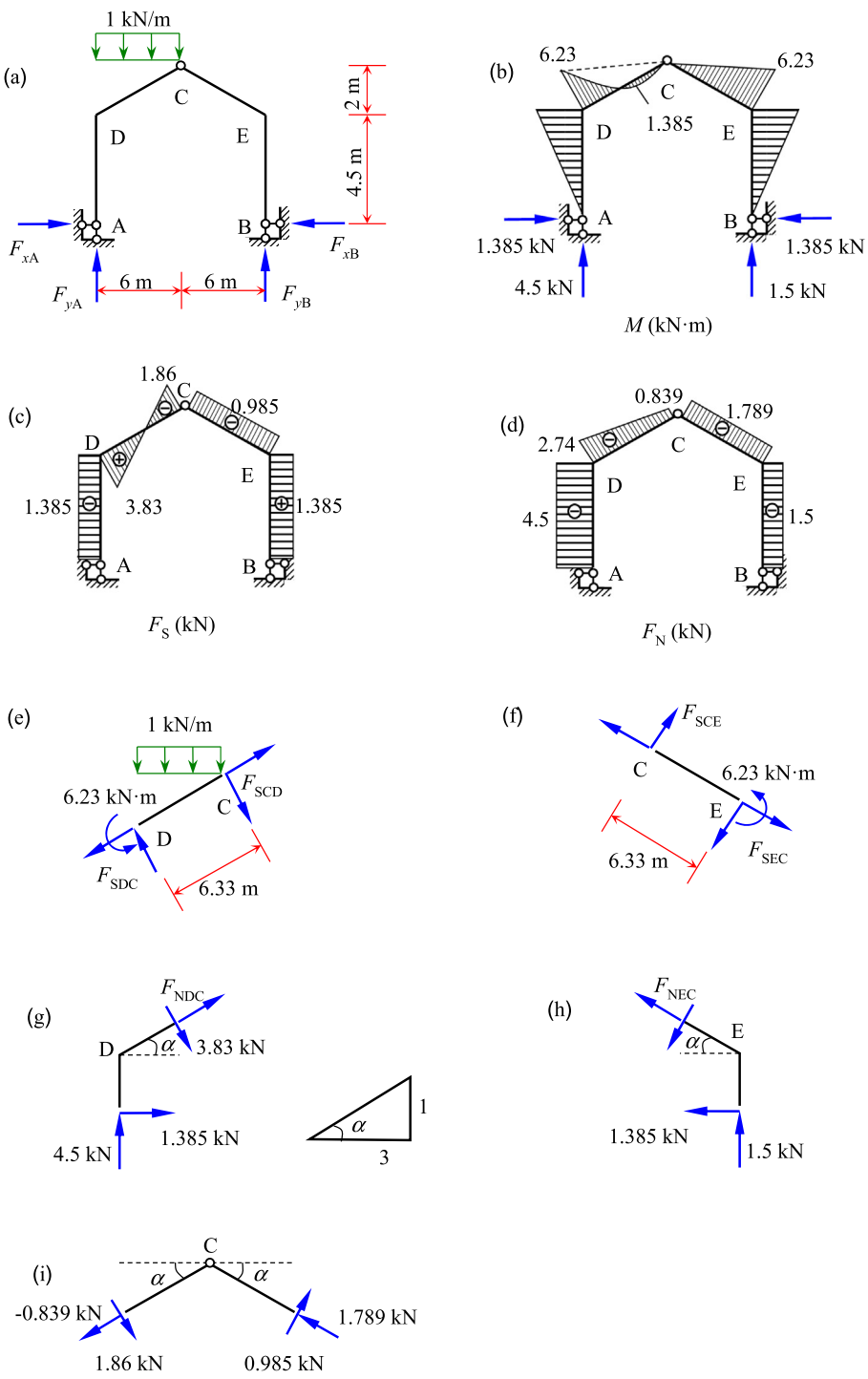


FIG. 3.33 – Portal frame for example 3.12.

For members AD and BE, one side of the section can be taken as a free body, and the shears of member ends can be obtained by the direct method as follows:

$$F_{\text{SAD}} = F_{\text{SDA}} = -1.385 \text{ kN}$$

$$F_{\text{SBE}} = F_{\text{SEB}} = 1.385 \text{ kN}$$

For members CD and CE, they can be taken as free bodies (figure 3.33e, f), and the indirect method can be used to solve the shears of member ends as follows:

$$F_{\text{SDC}} = \frac{1}{6.33 \text{ m}} \times (6.23 \text{ kN} \cdot \text{m} + 6 \text{ kN} \times 3 \text{ m}) = 3.83 \text{ kN}$$

$$F_{\text{SCD}} = \frac{1}{6.33 \text{ m}} \times (6.23 \text{ kN} \cdot \text{m} - 6 \text{ kN} \times 3 \text{ m}) = -1.86 \text{ kN}$$

$$F_{\text{SCE}} = F_{\text{SEC}} = \frac{1}{6.33 \text{ m}} \times (-6.23 \text{ kN} \cdot \text{m}) = -0.985 \text{ kN}$$

The shear diagram F_s is exhibited in figure 3.33c.

(4) Draw the axial force diagram F_N

Axial force at the member end can be solved with direct and indirect methods, respectively.

For members AD and BE, one side of the section can be taken as a free body, and the axial forces at member ends are obtained as follows:

$$F_{\text{NAD}} = F_{\text{NDA}} = -4.5 \text{ kN}$$

$$F_{\text{NBE}} = F_{\text{NEB}} = -1.5 \text{ kN}$$

As for the axial forces at the ends of members DC and CE, the joints can be used as free bodies. Taking joint D as a free body (figure 3.33g), and establishing the projection equation along the axis of member DC, F_{NDC} can be obtained as follows:

$$F_{\text{NDC}} + 1.384 \text{ kN} \cos \alpha + 4.5 \text{ kN} \sin \alpha = 0$$

$$F_{\text{NDC}} = -2.74 \text{ kN}$$

Similarly, from the free body diagram of joint E (figure 3.33h), the projection equation is taken with EC as the axis:

$$F_{\text{NEC}} = -1.384 \text{ kN} \cos \alpha - 1.5 \text{ kN} \sin \alpha = -1.789 \text{ kN}$$

Because there is no load on member EC along its axis direction, the axial force along the member is unchanged, that is,

$$F_{\text{NCE}} = -1.789 \text{ kN}$$

For solving F_{NCD} , the free body diagram shown in figure 3.33i can be used,

$$\sum F_x = 0, -F_{\text{NCD}} \cos \alpha + 1.86 \text{ kN} \sin \alpha + 0.985 \text{ kN} \sin \alpha - 1.789 \text{ kN} \cos \alpha = 0$$

$$F_{\text{NCD}} = -0.839 \text{ kN}$$

The axial force diagram F_N is displayed in figure 3.33d.

(5) Validation

Any part of the frame can be intercepted to check whether the equilibrium conditions are met. For example, the free body of joint C (figure 3.33i) can be used to validate $\sum F_y = 0$.

3.4.4 Quick Drawing of Moment Diagrams of Statically Determinate Frames

The internal force analysis of statically determinate frames is not only required by strength calculation but also the basis for displacement calculation and analysis of the statically indeterminate frames, especially the drawing of moment diagrams, which are widely used in the following sections and need to be thoroughly mastered. In some cases, the moment diagram of statically determinate frames can be drawn quickly and accurately according to the characteristics of the loads, which will get twice the result with half the effort. For example, for a cantilever frame, the moment diagram usually can be drawn directly without calculating the support reactions. For a three-hinged or master-slave frame, in general, it is not necessary to find all the support reactions, only the support reactions perpendicular to the member need to be solved, and the drawing of the moment diagram can begin from the support. In addition, the relations between the moment diagram and loads, supports and joints, as well as the structure symmetry, can be used to quickly draw the moment diagram.

Example 3.13. Draw the internal force diagram of the frame shown in figure 3.34a.

Solution: The figure shows a three-hinged frame. When drawing the moment diagram, it is not required to solve all the unknown support reactions, and only support reactions perpendicular to the member need to be calculated first, where the diagram can be drawn from. For the frame shown in the figure, because there are no loads on the left half of the frame and the moment at point C is zero, the action line of the support reaction of point A must pass through point C. Then, the moment equilibrium equation is established centered at point O, which is the intersection point of AC and BD, and the horizontal support reaction F_{xB} of point B can be obtained.

$$\sum M_O = 0, \quad F_{xB} \cdot 2a - M = 0, \quad F_{xB} = \frac{M}{2a}$$

Because F_{yB} does not generate moments on member BD, the moments of member BD can be determined by F_{xB} , and the moment diagram can be drawn as shown in figure 3.34b. Then, M_{DC} can be obtained based on the equilibrium condition of joint D. In addition, C is a hinge, so its moment is zero. There is a couple moment M at the right side of joint C, so M_{CD} equals to M and the upper side is in tension. Because there are no other loads on member CD, the moment diagram of member CD can be drawn by connecting M_{CD} and M_{DC} with a straight line. Then, the moment diagram of member CE can be determined according to the properties of the joints and the relations between loads and internal forces. The moment M_{CE} is

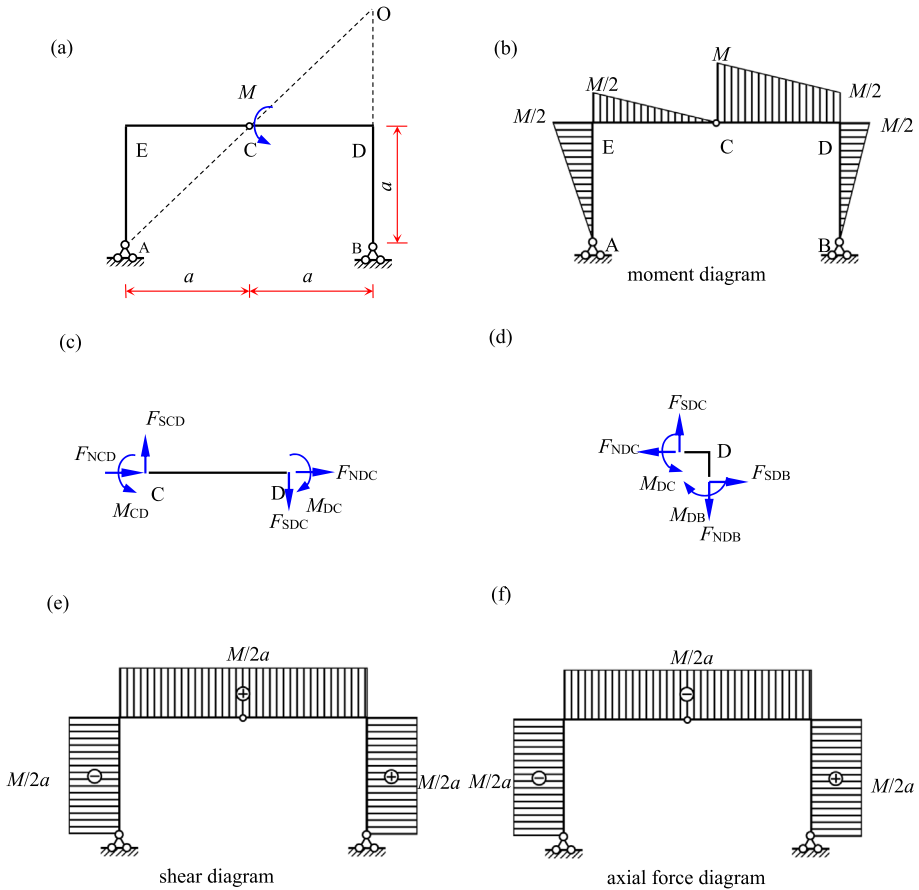


FIG. 3.34 – Three-hinged frame for example 3.13.

zero, and the moment diagram of member CE is a straight line and parallel to the moment diagram of member CD, so M_{EC} can be concluded as $M/2$. Finally, according to the equilibrium condition of joint E, $M_{EA} = M_{EC} = M/2$, and according to the properties of the pin support, it can be known that $M_{AE} = 0$, and the moment diagram of member AE can be obtained by line connecting.

After drawing the moment diagram of the frame, the indirect method can be used to draw the shear and axial force diagrams based on the moment diagram of the frame. The shear diagram of member BD can be drawn by F_{xB} . In order to draw the shear diagram of member DC, member DC can be taken as a free body, as shown in figure 3.34c. Taking moments with C as the center, then

$$\sum M_C = 0, \quad M - M_{DC} - F_{SDC} \cdot a = 0, \quad F_{SDC} = \frac{M}{2a}$$

From the equilibrium relation of joint D, as shown in figure 3.34d, then:

$$F_{\text{NDB}} = F_{\text{SDC}} = \frac{M}{2a}, \quad F_{\text{NDC}} = F_{\text{SDB}} = -F_{\text{xB}} = -\frac{M}{2a}$$

Then shear and axial force diagrams can be drawn, as shown in figure 3.34e, f.

Example 3.14. Quickly draw the moment diagram of the frame shown in figure 3.35a.

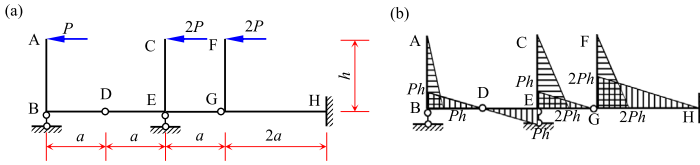


FIG. 3.35 – Master-slave frame for example 3.14.

Solution: In this master-slave frame, members AB, CE, and FG are all cantilever structures, so their moment diagrams can be drawn directly. Then according to the equilibrium condition of joints, M_{BD} can be determined, and M_{D} is zero, which can be connected and extended to determine the moment diagram of member BE. In the same way, according to the equilibrium conditions of joint E, M_{EG} can be determined, and the moment at the hinge of member EG is zero, so the moment diagram of member EG can be drawn. Finally, M_{GH} can be determined based on the equilibrium condition of joint G, and M_{GF} is equivalent to a couple moment acting at member EH, so the moment diagrams of the left and right sides of joint G should be parallel to each other, and the moment diagram of member GH is determined accordingly. The moment diagram of the frame is demonstrated in figure 3.35b.

3.5 Statically Determine Composite Structures

A composite structure refers to a structure composed of links and beams. Among them, links are only subjected to axial force, while beams are subjected to the moment, shear and axial force at the same time. Composite structures are often used in load-bearing structures such as roof trusses, crane beams, and bridges (as shown in figure 3.36). The internal forces of beams are mainly moments, and they are generally made of reinforced concrete. The internal forces of links are mainly tensile forces, and links are generally made of steel.

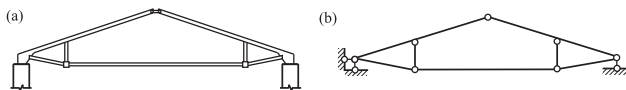


FIG. 3.36 – (a) Roof truss and (b) simplified calculation diagram.

1. Force analysis of statically determined composite structures

The basic principle of the force analysis of statically determined composite structures is the same as the one for general statically determined structures. Usually, the support reactions are first obtained, and then the axial force of each link is calculated. Finally, the internal forces of beams are analyzed. When the composite structure is analyzed and calculated with the method of sections, considering that the beam has not only the axial force, but also the moment and shear, it is necessary to avoid truncating beams. In addition, the statically determined composite structure generally has a key link. The axial force of the link can be obtained first with the method of sections.

Example 3.15. Perform the internal force analysis on the composite structure shown in figure 3.37a, calculate the axial forces of the links, and make the moment diagram of the beams. In the figure, $a = 4$ m, $h = 3$ m, $q = 15$ kN/m, $F_P = 30$ kN.

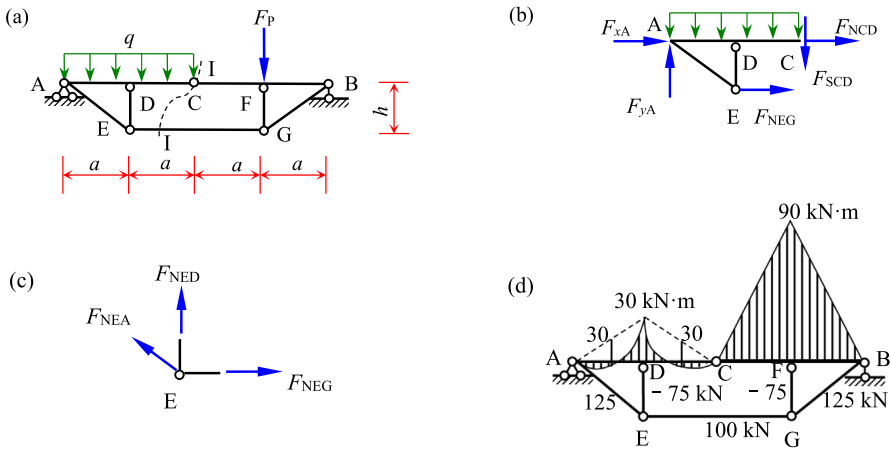


FIG. 3.37 – Composite structure for example 3.15.

Solution: Firstly, solve the support reactions according to the whole equilibrium equations.

$$\sum F_x = 0, \quad F_{xA} = 0$$

$$\sum M_B = 0, \quad F_{yA} \cdot 4a - q \cdot 2a \cdot 3a - F_P \cdot a = 0, \quad F_{yA} = 97.5 \text{ kN}$$

$$\sum F_y = 0, \quad F_{yA} + F_{yB} = q \cdot 2a + F_P, \quad F_{yB} = 52.5 \text{ kN}$$

Then, in order to avoid truncating the beam, take section I-I along point C to divide the composite structure into two parts. Take the left half as a free body, and

establish the moment equilibrium equation with point C as the center to obtain the axial force F_{NEG} of the key link EG, as shown in figure 3.37b.

$$\sum M_C = 0, \quad F_{yA} \cdot 2a - q \cdot 2a \cdot a - F_{\text{NEG}} \cdot h = 0, \quad F_{\text{NEG}} = 100 \text{ kN (tension)}$$

$$\sum F_x = 0, \quad F_{\text{NCD}} = -F_{\text{NEG}}, \quad F_{\text{NCD}} = -100 \text{ kN (compression)}$$

$$\sum F_y = 0, \quad F_{\text{SCD}} + q \cdot 2a = F_{yA}, \quad F_{\text{SCD}} = -22.5 \text{ kN}$$

$$M_{\text{DC}} = -qa \cdot \frac{a}{2} - F_{\text{SCD}} \cdot a = -30 \text{ kN} \cdot \text{m (the upper side is in tension)}$$

Take joint E as a free body, as shown in figure 3.37c, then

$$F_{x\text{NEA}} = F_{\text{NEG}}, \quad F_{x\text{NEA}} = 100 \text{ kN}$$

Therefore, $F_{\text{NEA}} = 125 \text{ kN}$, $F_{y\text{NEA}} = 75 \text{ kN}$, then

$$F_{\text{NED}} = -F_{y\text{NEA}}, \quad F_{\text{NED}} = -75 \text{ kN (compression)}$$

In the same way, take the right side of point C as a free body, then

$$F_{\text{NCF}} = -100 \text{ kN}$$

$$F_{\text{SCF}} = -22.5 \text{ kN}$$

$$F_{\text{NGB}} = 125 \text{ kN}$$

$$F_{\text{NGF}} = -75 \text{ kN}$$

$$M_{\text{FC}} = F_{\text{SCF}} \cdot a = -90 \text{ kN} \cdot \text{m (the upper side is in tension)}$$

Finally, the internal force diagram of the structure is shown in figure 3.37d.

Example 3.16. Draw the internal force diagram of the bottom-supported pentagonal composite roof truss shown in figure 3.38a.

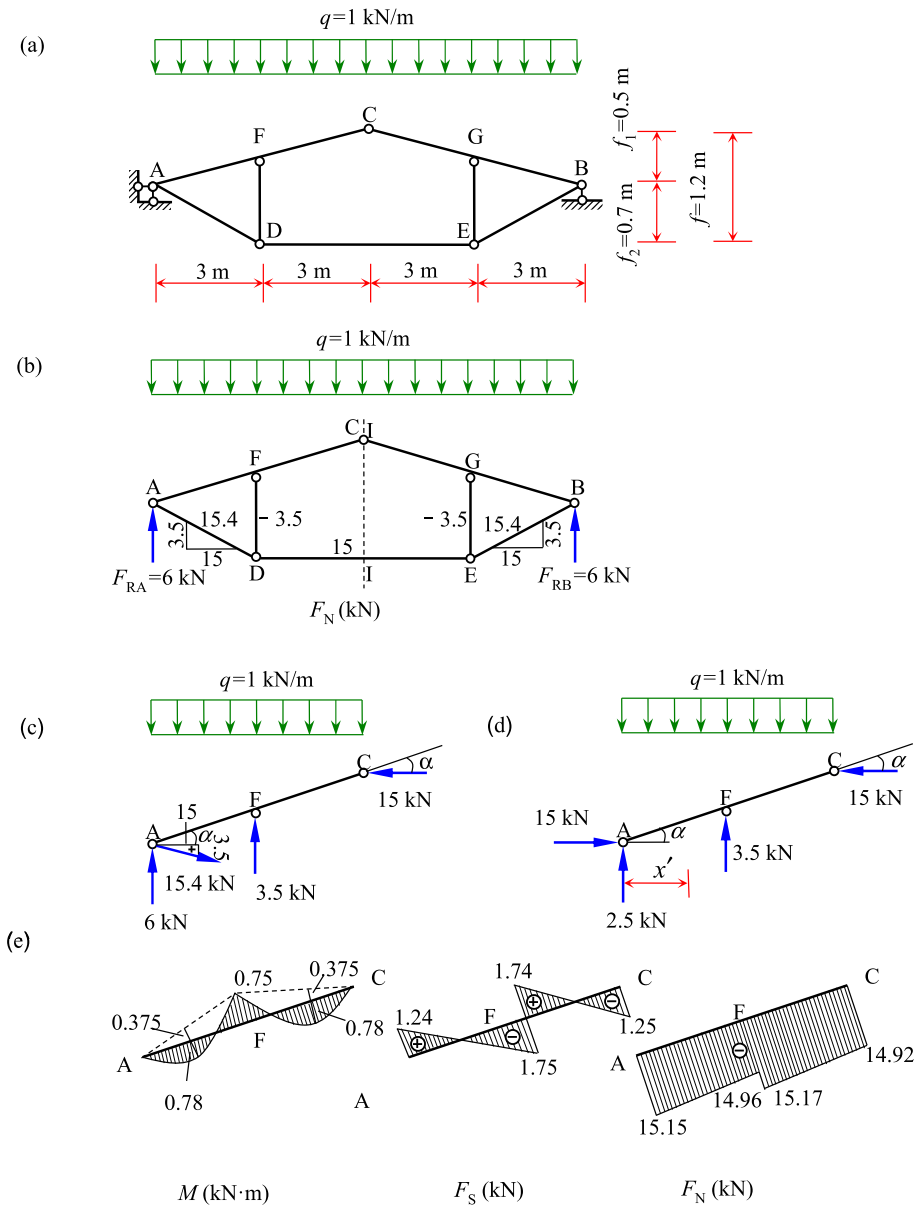
Solution: (1) Calculation of the internal forces of the links

This structure is made up of the left and right parts ACD and BCE, which have no redundant constraints and are connected with hinge C and link DE. Thus, the structure is statically determined. When calculating the internal forces, section I-I can be made first to cut hinge C and link DE (figure 3.38b). In terms of the moment equilibrium equation $\sum M_c = 0$, then we have

$$6 \text{ kN} \times 6 \text{ m} - 1 \text{ kN/m} \times 6 \text{ m} \times 3 \text{ m} - 1.2 F_{\text{NDE}} = 0$$

Therefore

$$F_{\text{NDE}} = 15 \text{ kN}$$



From joints D and E, the axial forces of all the links can be obtained. The calculation results are shown in figure 3.38b.

(2) Internal force diagrams of the beams

Loads of member AFC are shown in figure 3.38c. After combining the vertical forces on joint A, the force diagram is shown in figure 3.38d. The internal force diagram is obtained as shown in figure 3.38e. The corresponding description is presented as follows.

The shear and axial force of any section can be calculated based on formulas as follows:

$$\begin{aligned} F_S &= F_y \cos \alpha - 15 \sin \alpha \\ F_N &= -F_y \sin \alpha - 15 \cos \alpha \end{aligned}$$

where F_y is the resultant of the vertical forces on the section. In the figure, $\sin \alpha = 0.0835$ and $\cos \alpha = 0.996$. Therefore, the shear and axial forces at end A are

$$\begin{aligned} F_{SAF} &= 2.5 \text{ kN} \cos \alpha - 15 \text{ kN} \sin \alpha \\ &= 2.5 \text{ kN} \times 0.996 - 15 \text{ kN} \times 0.0835 \\ &= 1.24 \text{ kN} \end{aligned}$$

$$\begin{aligned} F_{NAF} &= -2.5 \text{ kN} \sin \alpha - 15 \text{ kN} \cos \alpha \\ &= -2.5 \text{ kN} \times 0.0835 - 15 \text{ kN} \times 0.996 \\ &= -15.15 \text{ kN} \end{aligned}$$

The maximum moment in the span occurs when the shear is zero, and its horizontal coordinate is represented by x' (figure 3.38d). By

$$F_S = F_y \cos \alpha - 15 \text{ kN} \sin \alpha = 0$$

Then

$$F_y = 15 \text{ kN} \tan \alpha, \text{ in section AF, } F_y = 2.5 \text{ kN} - qx'$$

That is

$$2.5 \text{ kN} - qx' = 15 \text{ kN} \tan \alpha$$

Then

$$x' = \frac{2.5 \text{ kN}}{1 \text{ kN/m}} - \frac{15 \text{ kN}}{1 \text{ kN/m}} \times \frac{0.5 \text{ m}}{6 \text{ m}} = 1.25 \text{ m}$$

Therefore, the maximum moment is

$$\begin{aligned} M_{x'} &= 2.5 \text{ kN} \times 1.25 \text{ m} - 15 \text{ kN} \times \left(\frac{0.5 \text{ m}}{6 \text{ m}} \times 1.25 \text{ m} \right) \\ &\quad - \left(1 \text{ kN/m} \times 1.25 \text{ m} \times 1.25 \text{ m} \times \frac{1}{2} \right) \\ &= 0.78 \text{ kN} \cdot \text{m} \end{aligned}$$

2. Force characteristics of composite roof trusses

Example 3.16 and three cases as shown in figure 3.39a–c ($f = 1.2$ m, while f_1/f_2 is different) are discussed below.

There are two main factors affecting the internal force state of the under-supported pentagonal composite roof truss:

(1) Ratio of height to span f/l

The axial force F_{NDE} can be calculated using the thrust formula of the three-hinged arch (see equation (3.10) in the next section):

$$F_{NDE} = \frac{M_C^0}{f}$$

The axial force F_{NDE} of the lower chord DE is inversely proportional to f (the smaller the ratio of height to span, the larger the axial force F_{NDE}), and it has nothing to do with f_1 and f_2 (in figure 3.39, although f_1/f_2 is different, F_{NDE} has the same value).

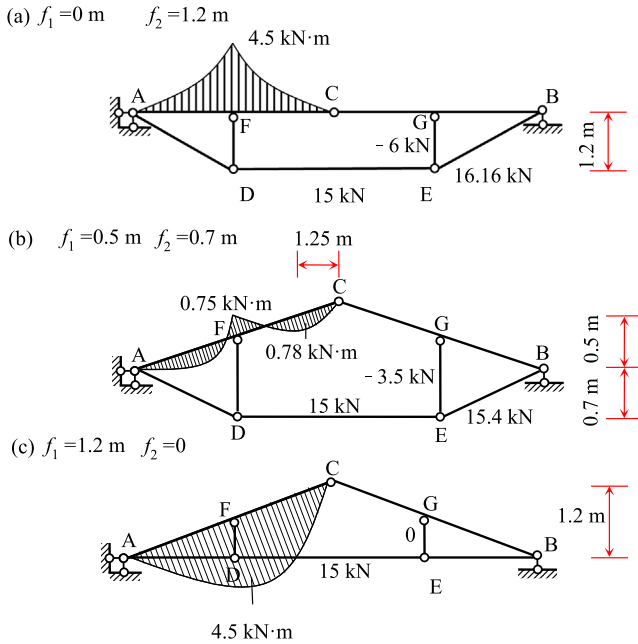


FIG. 3.39 – Three cases of composite structure.

(2) Ratio f_1/f_2

In figure 3.39, since F_{NDE} has the same value, the change of the axial forces of other chords is not large, while the change of the moments of the upper chords is large.

When $f_1 = 0$, the slope of the upper chord is zero, it is an under-supported composite structure with parallel chords (figure 3.39a). Herein, all the upper chords have negative moments, and the upper chord AFC is equivalent to a cantilever beam supported at points A and F.

When $f_2 = 0$, it is a three-hinged arch roof truss with tie rods (figure 3.39c). Herein, all the upper chords have positive moments, and the upper chord AFC is like a simply supported beam supported at points A and C.

When $f_1 = 0.5$ m and $f_2 = 0.7$ m, the negative moment at joint F of the upper chord and the maximum positive moment within two spans are approximately equal in value (figure 3.39b), and that value is much smaller than that of two limited cases in figure 3.39a, c.

3.6 Three-Hinged Arches

In the construction of houses, hydraulic structures, and bridges, arches are a very old structure form but are still widely used at present. The characteristic of the arch structure is that horizontal thrust will arise at the arch support under vertical loads. Therefore, the arch is also called a thrust structure. Due to the thrust of the arch support, the arch is mainly subjected to the axial compression, while the moment and shear on the section are small. This force characteristic makes an arch suitable to be constructed using relatively cheap natural materials with high compressive strength and low tensile strength, such as brick, stone, and concrete, so arches have been widely used since ancient times.

There are many types of arches, which can be divided into three-hinged arches without redundant constraints (figure 3.40a, b), and two-hinged and hingeless arches with redundant constraints (figure 3.40c, d) based on the geometrical construction.

The **three-hinged arch** is a statically-determined arch structure. Due to the existence of horizontal thrust, the three-hinged arch shown in figure 3.40a has certain requirements on the base. When the horizontal resistance of the base is weak, an arch with a horizontal tie rod should be used, as shown in figure 3.40b. Herein, the horizontal thrust of the support will be provided by the tie rod.

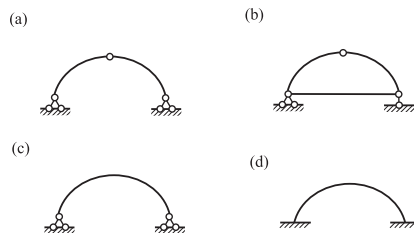


FIG. 3.40 – (a, b) Three-hinged arch, (c) two-hinged arch, and (d) hingeless arch.

The axis of the three-hinged arch is usually a parabola or arc, and sometimes a catenary is also used. In practical applications, a suitable axis form should be selected according to the characteristics of the loads. The mechanical properties of the arch are closely related to the ratio of arch height f to span l , so the ratio of height to span f/l is its basic parameter (shown in figure 3.41). In engineering applications, the ratio of height to the span of the arch is usually between 1 and 1/10.

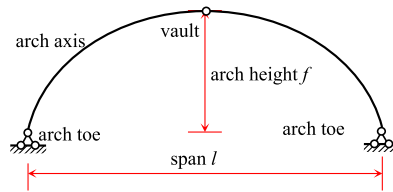


FIG. 3.41 – Three-hinged arch.

3.6.1 Support Reactions and Internal Force Calculation of Three-Hinged Arches

Take the three-hinged arch under vertical loads shown in figure 3.42a as an example to illustrate the calculation process of the support reactions and internal forces. In order to facilitate the comparison of the mechanical performance between arches and beams, figure 3.42b displays a reference simply supported beam with the same span, loads, and action locations as a three-hinged arch, and the force analysis is performed simultaneously.

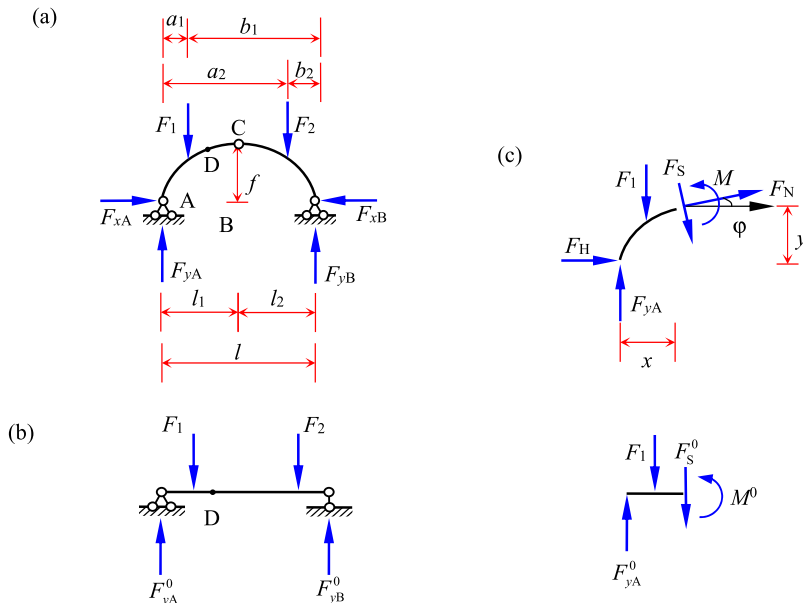


FIG. 3.42 – Support reactions and internal force calculation of three-hinged arches.

The support reactions of the three-hinged arch are analyzed, and the whole equilibrium of the three-hinged arch is considered first. As shown in figure 3.42a, the arch is treated as a free body, and the moment equilibrium equations are formulated at points A and B, respectively. The vertical support reactions at points A and B can be obtained:

$$\begin{aligned}\sum M_A = 0, \quad F_{yB} &= \frac{1}{l}(F_1 a_1 + F_2 a_2) \\ \sum M_B = 0, \quad F_{yA} &= \frac{1}{l}(F_1 b_1 + F_2 b_2)\end{aligned}\quad (3.8)$$

Similarly, the vertical support reactions of the reference simply supported beam can be obtained, and

$$\begin{aligned}F_{yA} &= F_{yA}^0 \\ F_{yB} &= F_{yB}^0\end{aligned}\quad (3.9)$$

In addition, from the horizontal projection equilibrium equation of the free body, it can be concluded as:

$$F_{xA} = F_{xB} = F_H$$

Herein, F_H represents the horizontal support reaction of the three-hinged arch under loads, *i.e.*, the thrust force. In order to solve F_H , the left half of the arch can be taken as a free body, and the moment equilibrium equation of point C is established, then

$$\begin{aligned}\sum M_C = 0, \quad F_{yA} \cdot l_1 - F_1 \cdot (l_1 - a_1) - F_H \cdot f &= 0 \\ F_H &= \frac{F_{yA} \cdot l_1 - F_1 \cdot (l_1 - a_1)}{f}\end{aligned}$$

Then analyze the moment of section C of the reference beam at the corresponding position of the top hinge of the three-hinged arch.

$$M_C^0 = F_{yA}^0 \cdot l_1 - F_1 \cdot (l_1 - a_1)$$

Therefore, the thrust of the three-hinged arch is

$$F_H = \frac{M_C^0}{f}\quad (3.10)$$

It can be seen that the vertical support reaction of the three-hinged arch is the same as that of the reference beam. The thrust F_H is equal to the ratio of the moment of section C of the reference beam to the arch height f . Its value is only related to the locations of three hinges and has nothing to do with the curve shape of the arch axis among hinges, *i.e.*, only related to the ratio of height to span f/l of the arch. When the loads and arch span are constant, the thrust F_H is inversely proportional to arch height f , *i.e.*, the lower the arch, the greater the thrust. If the arch height approaches 0, the thrust tends to infinity, and the three hinges A, B, and C are on one straight line such that the arch becomes a geometrically changeable system.

After the support reactions are obtained, the internal forces of any section D of the arch can be obtained by the method of sections. However, when selecting the section, it should be noted that the arch axis is a curve. The section should be orthogonal to the arch axis, *i.e.*, perpendicular to the tangent of the arch axis. The moment of the section is M , the shear is F_S (along the direction of the section), and the axial force is F_N (perpendicular to the section), as shown in figure 3.42c. The moment and shear at the corresponding section D of the reference beam are marked as M^0 and F_S^0 , respectively.

Solve the moment of the section of the arch first, taking the left side of section D as a free body, and establish the moment equilibrium equation, then

$$\sum M_D = F_{yA} \cdot x - F_H \cdot y - F_1 \cdot (x - a_1) - M = 0$$

The moment of section D of the arch can be determined

$$M = F_{yA} \cdot x - F_1 \cdot (x - a_1) - F_H \cdot y$$

And

$$\begin{aligned} F_{yA} &= F_{yA}^0 \\ M^0 &= F_{yA}^0 \cdot x - F_1 \cdot (x - a_1) \end{aligned}$$

Then

$$M = M^0 - F_H \cdot y \quad (3.11)$$

The moment M is positive when the inside of the arch is in tension. It can be seen that the moment of any section of the three-hinged arch is equal to that of the corresponding section of the reference beam minus $F_H \cdot y$ caused by the thrust of the arch. The moment is smaller than that of the reference beam because of the thrust.

Then calculate the shear and axial forces of the section. The shear is positive when it makes the free body rotate clockwise, and axial force is positive when the free body is in tension. Define the angle between the tangent of the arch axis and the horizontal line as φ . φ is positive in the left half of the arch, and is negative in the right half. Establish the projection equilibrium equations of the free body in the direction of the shear and axial force of section D

$$\begin{aligned} F_S + F_1 \cdot \cos \varphi + F_H \cdot \sin \varphi - F_{yA} \cdot \cos \varphi &= 0 \\ F_N + F_H \cdot \cos \varphi + F_{yA} \cdot \sin \varphi - F_1 \cdot \sin \varphi &= 0 \end{aligned}$$

And

$$F_S^0 = F_{yA}^0 - F_1$$

Then

$$\begin{aligned} F_S &= F_S^0 \cdot \cos \varphi - F_H \cdot \sin \varphi \\ F_N &= -F_S^0 \cdot \sin \varphi - F_H \cdot \cos \varphi \end{aligned} \quad (3.12)$$

Combined with the moment calculation equation of the section, the internal force calculation formulas of any section of the three-hinged arch are obtained. It can be known from the equation that the internal forces of the three-hinged arch are not only related to the locations of the three hinges but also to the axis shape of the arch. Moreover, the axial force of the three-hinged arch is relatively large, which is generally compression.

Example 3.17. Draw the internal force diagrams of the three-hinged arch shown in figure 3.43a. When the left support is taken as the origin of coordinates, its arch axis is a parabola, and the equation is:

$$y = \frac{4f}{l^2} x(l - x)$$

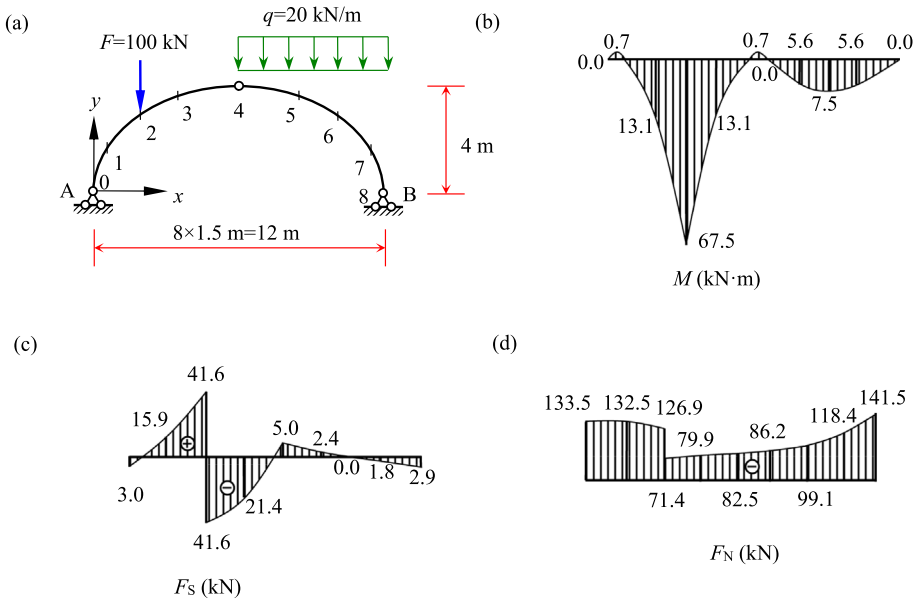


FIG. 3.43 – Three-hinged arch for example 3.17.

Solution: Solve the support reactions first. According to equations (3.9) and (3.10)

$$F_{yA} = F_{yA}^0 = \frac{100 \times 9 + 20 \times 6 \times 3}{12} = 105 \text{ kN}$$

$$F_{yB} = F_{yB}^0 = \frac{100 \times 3 + 20 \times 6 \times 9}{12} = 115 \text{ kN}$$

$$F_H = \frac{M_C^0}{f} = \frac{105 \times 6 - 100 \times 3}{4} = 82.5 \text{ kN}$$

After the support reactions are obtained, the internal force diagrams of the three-hinged arch can be drawn according to the internal force equations (3.11) and (3.12). To this end, the arch span is divided into eight equal parts, and the internal forces on the section of each equal-part point are obtained. The specific values are shown in table 3.1. Then, the internal force diagrams of the three-hinged arch can be drawn according to the values in the table, as shown in figure 3.43b–d.

Now, sections 1 and 2 are taken as examples to describe the internal force calculation process of the section.

At section 1, $x_1 = 1.5 \text{ m}$, based on the arch axis equation:

$$y_1 = \frac{4f}{l^2} x_1(l - x_1) = \frac{4 \times 4}{12^2} \times 1.5 \times (12 - 1.5) = 1.75 \text{ m}$$

The tangent slope at section 1 is:

$$\tan \varphi_1 = \left(\frac{dy}{dx} \right)_1 = \frac{4f}{l^2} (l - 2x_1) = \frac{4 \times 4}{12^2} (12 - 2 \times 1.5) = 1$$

So the tangential angle is $\varphi_1 = 45^\circ$, and

$$\sin \varphi_1 = \cos \varphi_1 = 0.707$$

According to the internal force calculation equations, the moment, shear, and axial force of the section are obtained as follows:

$$M_1 = M_1^0 - F_H y_1 = 105 \times 1.5 - 82.5 \times 1.75 = 13.1 \text{ kN}\cdot\text{m}$$

$$F_{S1} = F_{S1}^0 \cos \varphi_1 - F_H \sin \varphi_1 = 105 \times 0.707 - 82.5 \times 0.707 = 15.9 \text{ kN}$$

$$F_{N1} = -F_{S1}^0 \sin \varphi_1 - F_H \cos \varphi_1 = -105 \times 0.707 - 82.5 \times 0.707 = -132.5 \text{ kN}$$

At section 2, due to the concentrated load, the shear and axial forces on the left and right sides of the section are abruptly changed, so they should be solved separately:

$$F_{S2L} = F_{S2L}^0 \cos \varphi_2 - F_H \sin \varphi_2 = 105 \times 0.832 - 82.5 \times 0.555 = 41.6 \text{ kN}$$

$$F_{S2R} = F_{S2R}^0 \cos \varphi_2 - F_H \sin \varphi_2 = 5.0 \times 0.832 - 82.5 \times 0.555 = -41.6 \text{ kN}$$

$$F_{N2L} = -F_{S2L}^0 \sin \varphi_2 - F_H \cos \varphi_2 = -105 \times 0.555 - 82.5 \times 0.832 = -126.9 \text{ kN}$$

$$F_{N2R} = -F_{S2R}^0 \sin \varphi_2 - F_H \cos \varphi_2 = -5.0 \times 0.555 - 82.5 \times 0.832 = -71.4 \text{ kN}$$

The internal force calculation of other sections is similar to the above process. Finally, it should be pointed out that in figure 3.43c, the shear of one section in segment 0–1 and segment 3–4 is 0, and the minimum moments of these two sections can be obtained, which both are $0.7 \text{ kN}\cdot\text{m}$, as shown in figure 3.43b.

TAB. 3.1 – Internal force calculation of the three-hinged arch.

	Section	0	1	2	3	4	5	6	7	8	
Geometric parameters and shear of reference beam	y (m)	0.0	1.75	3.0	3.75	4.0	3.75	3.0	1.75	0.0	
	$\tan\varphi$	1.333	1.000	0.667	0.333	0.000	-0.333	-0.667	-1.000	-1.333	
	$\sin\varphi$	0.800	0.707	0.555	0.316	0.000	-0.316	-0.555	-0.707	-0.800	
	$\cos\varphi$	0.599	0.707	0.832	0.948	1.000	0.948	0.832	0.707	0.599	
	F_S^0 (kN)	L R	105.0	105.0	105.0 5.0	5.0	5.0	-25.0	-55.0	-85.0	-115.0
Moment (kN·m)	M_0		0.00	157.0	315.0	322.5	330.0	315.0	255.0	150.0	0.00
	$-F_H y$		0.00	-144.4	-247.5	-309.4	-330.0	-309.4	-247.5	-144.4	0.00
	M		0.00	13.1	67.5	13.1	0.00	5.6	7.5	5.6	0.00
Shear (kN)	$F_S^0 \cos\varphi$	L R	63.0	74.2	87.4 4.2	4.7	5.0	-23.7	-45.8	-60.1	-68.9
	$-F_H \sin\varphi$		-66.0	-58.3	-45.8	-26.1	0.00	26.1	45.8	58.3	66.0
	F_S	L R	-3.0	15.9	41.6 -41.6	-21.4	5.0	2.4	0.00	-1.8	-2.9
Axial force (kN)	$-F_S^0 \sin\varphi$	L R	-84.0	-74.2	-58.3 -2.8	-1.6	0.00	-7.9	-30.5	-60.1	-92.0
	$-F_H \cos\varphi$		-49.5	-58.3	-68.6	-78.3	-82.5	-78.3	-68.6	-58.3	-49.5
	F_N	L R	-133.5	-132.5	-126.9 -71.4	-79.9	-82.5	-86.2	-99.1	-118.4	-141.5

3.6.2 Rational Axes of Three-Hinged Arches

Because of the horizontal support thrust of the three-hinged arch, its moments of sections are greatly reduced compared with those of the reference beam. However, under normal circumstances, the three-hinged arch has the moment, shear and axial force on the section, which is in an acentric compression state, and its normal stress distribution is non-uniform. In the most ideal case, an appropriate arch axis can be designed so that under the action of a fixed load, all the sections of the arch only bear compressive force, and their moments are all zero. Herein, the normal stress distribution on any section will be uniform, and the material is most fully and economically used. The axis that makes the arch in a no-moment state under fixed loads is called a **rational axis of arch**, in which the pressure line of the arch (*i.e.*, the pressure polygon obtained with the graphic method) coincides with the arch axis.

It can be known from equation (3.11) that under vertical loads, the moment of any section of the three-hinged arch is:

$$M = M^0 - F_H \cdot y$$

This equation shows that the moment of the three-hinged arch is obtained by superposing the moment M^0 of the reference beam on $-F_H \cdot y$. When the arch span and loads have been determined, M^0 does not change with the arch axis. While F_H is only related to the locations of the three hinges, and has nothing to do with the axis, but y is related to the arch axis. Therefore, the axis form of the arch should be properly selected between the three hinges, so that any section of the arch is in a state without moments, that is:

$$M = M^0 - F_H \cdot y = 0$$

So

$$y = \frac{M^0}{F_H} \quad (3.13)$$

According to the above equation, under the action of vertical loads, the ordinate y of the rational axis of the three-hinged arch is proportional to the moment of the reference simply supported beam. When the vertical load on the three-hinged arch is known, only the moment equation of the reference beam is required, and then divided by the horizontal reaction F_H , the rational axis equation of the arch can be obtained.

Example 3.18. Solve the rational axis function when the symmetrical three-hinged arch is subjected to a uniformly distributed vertical load shown in figure 3.44a.

Solution: The moment equation of the reference simply supported beam (figure 3.44b) is

$$M^0 = \frac{q}{2}x(l-x)$$

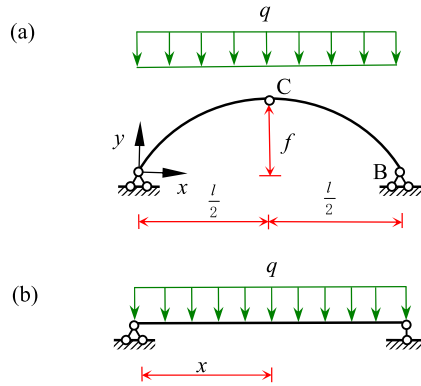


FIG. 3.44 – Symmetrical three-hinged arch under uniformly distributed vertical load.

The thrust of the arch can be obtained from equation (3.10) as

$$F_H = \frac{M_C^0}{f} = \frac{ql^2}{8f}$$

According to equation (3.13), the rational axis function of the arch can be obtained as

$$y = \frac{4f}{l^2} x(l - x)$$

It can be seen that, when the full span of the three-hinged arch is under the vertical uniformly distributed load, its rational axis is a parabola. In this equation, the arch height f is an unknown quantity. A set of parabolas with different ratios of height to span are all rational axes of the arch.

Example 3.19. The three-hinged arch is subjected to uniform water pressure. Prove that its rational axis is an arc (figure 3.45a).

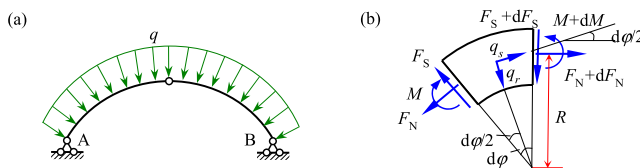


FIG. 3.45 – Three-hinged arch subjected to uniform water pressure.

Proof. The three-hinged arch is subjected to a uniform water pressure perpendicular to the arch axis, so equation (3.13) cannot be applied directly to solve the rational arch axis. Based on the equilibrium conditions, the differential relations between the distributed load and the internal force of the curved member are derived, and then the proof is given.

Take a differential element from the curved member as a free body, as shown in figure 3.45b. Let the curvature radius of the differential element be R , the angle between the cross-sections at both ends be $d\varphi$, and then the length of the differential element is $ds = R d\varphi$. Let s and r denote the tangent and normal directions of member axis, respectively. The load intensities along directions s and r are q_s and q_r .

From $\sum F_s = 0$, then

$$(F_N + dF_N) \cos \frac{d\varphi}{2} - F_N \cos \frac{d\varphi}{2} - (F_S + dF_S) \sin \frac{d\varphi}{2} - F_S \sin \frac{d\varphi}{2} + q_s ds = 0$$

Because $d\varphi$ is very small, let $\cos \frac{d\varphi}{2} \approx 1$, $\sin \frac{d\varphi}{2} \approx \frac{d\varphi}{2}$, and ignore the high-order terms, then

$$dF_N - F_S d\varphi + q_s ds = 0$$

Divided by ds , considering $ds = R d\varphi$, then

$$\frac{dF_N}{ds} = \frac{F_S}{R} - q_s$$

In the same way, from $ds = R d\varphi$, then

$$dF_S + F_N d\varphi + q_r ds = 0$$

That is $\frac{dF_S}{ds} = -\frac{F_N}{R} - q_r$.

Again, from $\sum M = 0$, then

$$dM - F_S ds = 0$$

That is

$$\frac{dM}{ds} = F_S$$

Taken together

$$\left. \begin{aligned} \frac{dF_N}{ds} &= \frac{F_S}{R} - q_s \\ \frac{dF_S}{ds} &= -\frac{F_N}{R} - q_r \\ \frac{dM}{ds} &= F_S \end{aligned} \right\} \quad (3.14)$$

Equation (3.14) is the differential relations between loads and the internal forces of the curved member. When $R \rightarrow \infty$, the curved member becomes a straight member, equation (3.14) becomes the equation of a straight member.

In this example, because the arch is subjected to a uniform water pressure q , the tangential load $q_s = 0$, and the normal load $q_r = q$ (constant). Therefore, the differential relation equation (3.14) of the internal force of a curved member can be written as:

$$\begin{cases} \frac{dF_N}{ds} = \frac{F_S}{R} & (a) \\ \frac{dF_S}{ds} = -\frac{F_N}{R} - q & (b) \\ \frac{dM}{ds} = F_S & (c) \end{cases}$$

If the arch is in the no-moment state, *i.e.*, $M = 0$, substituting this formula into formula (c), then

$$F_S = 0 \quad (d)$$

Substitute formula (d) into formula (a), and then $\frac{dF_N}{ds} = 0$, so we have

$$F_N = C \text{ (constant)} \quad (e)$$

Substitute formula (d) into formula (b), and then $0 = -\frac{F_N}{R} - q$, so

$$R = -\frac{F_N}{q} \quad (3.15)$$

It is known from formula (e) that the axial force of each section is a constant, and the load q is also a constant. Therefore, the curvature radius R of each segment should also be a constant. This means that the arch axis should be an arc.

It can be seen that under uniform water pressure, the rational arch axis is an arc, and the axial force is a constant. Therefore, circular sections are often applied in water pipes, high-pressure tunnels, and arch dams.

Example 3.20. Determine the rational axis of the symmetrical three-hinged arch under the soil-filling weight shown in figure 3.46. Take the density of the soil-filling be ρ and the vertical distributed load on the arch be $q = q_c + \rho g y$.

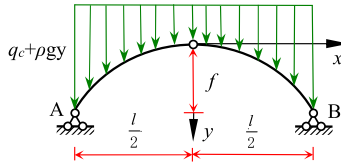


FIG. 3.46 – Symmetrical three-hinged arch under the soil-filling weight.

Solution: The characteristic of this example is that the intensity of the distributed load is related to the shape of the arch axis. It is a design-dependent load (that is, the design-related load), so the moment M^0 of the reference simply supported beam cannot be obtained in advance, and equation (3.13) cannot be used directly to find

the rational axis of the arch. Therefore, equation (3.13) is differentiated twice for x , then

$$\frac{d^2 y}{dx^2} = \frac{1}{F_H} \frac{d^2 M^0}{dx^2}$$

When q takes the downward direction as positive, according to the relations between loads and the internal forces of the simply supported beam, then

$$\frac{d^2 M^0}{dx^2} = -q(x)$$

So

$$\frac{d^2 y}{dx^2} = -\frac{q(x)}{F_H} \quad (3.16)$$

This is the differential equation of the rational axis of the arch under vertical loads. In the equation, y takes the upward direction as positive. In figure 3.46, the positive direction of the y -axis is downward, so the right side of equation (3.16) should take the positive sign, that is

$$\frac{d^2 y}{dx^2} = \frac{q(x)}{F_H} \quad (a)$$

Substitute $q = q_c + \rho g y$ into formula (a), then

$$\frac{d^2 y}{dx^2} - \frac{\rho g}{F_H} y = \frac{q_c}{F_H}$$

This is a linear second-order non-homogeneous differential equation with constant coefficients, and its solution can be represented by a hyperbolic function:

$$y = A \cdot \cosh \sqrt{\frac{\rho g}{F_H}} x + B \cdot \sinh \sqrt{\frac{\rho g}{F_H}} x - \frac{q_c}{\rho g}$$

The two constants A and B can be obtained from the boundary conditions:

$$\text{At } x = 0, y = 0, \text{ then } A = \frac{q_c}{\rho g}$$

$$\text{At } x = 0, \frac{dy}{dx} = 0, \text{ then } B = 0$$

So

$$y = \frac{q_c}{\rho g} \left(\cosh \sqrt{\frac{\rho g}{F_H}} x - 1 \right) \quad (b)$$

The above formula shows that the rational axis of the three-hinged arch is a catenary under the weight of soil filling.

It can be seen from the above examples that the three-hinged arch has different rational axes under different loads. Therefore, a rational axis determined based on fixed loads cannot guarantee the arch in a no-moment state under various loads.

The force state of the arch should be as close to the no-moment state as possible in the design, so the rational axis under the main load is usually used as the arch axis. In this way, the arch will only have small moments under general loads.

3.7 General Properties of Statically Determinate Structures

In terms of geometrical construction, statically determinate structures are geometrically unchangeable systems without redundant constraints. In terms of static equilibrium, all the internal forces and deformations of a statically determinate structure can be determined only by the equilibrium conditions. In other words, within the scope of linear elasticity, the support reactions and internal forces of the statically determinate structure satisfying all the equilibrium conditions are unique. The uniqueness of the internal forces that satisfy the equilibrium conditions is the basic static properties of statically determinate structures, from which several derived properties can be concluded.

- (1) Non-load factors such as temperature changes, support movements, and manufacturing errors do not cause internal forces in statically determinate structures

As shown in figure 3.47a, when the upper and lower sides of the cantilever beam are subjected to inconsistent temperature change, it can deform freely (as shown by the dashed line), so no internal forces are generated in the beam. Figure 3.47b demonstrates that when the support settlement of a simply supported beam occurs, it will only cause rigid body displacement of the member (as shown by the dashed line). Because there is no load on the simply supported beam, all equilibrium conditions can be satisfied when the internal forces and support reactions are all zero. According to the uniqueness of the solution, this is the true solution of the simply supported beam.

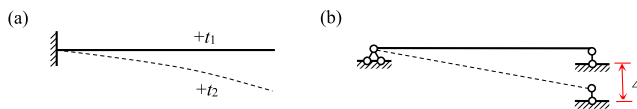


FIG. 3.47 – Statically determinate structures under (a) temperature change and (b) support movement.

- (2) Local equilibrium characteristic of the statically determinate structure

The statically determined structure only has internal forces in the local minimal geometrically unchangeable system under the action of an equilibrium force system, and other parts of the structure do not have internal forces and deformation.

As shown in figure 3.48a–c, the statically determined frame, truss, and three-hinged arch are all under the action of a set of balance forces. Through internal force analysis, it can be determined that the structure only has internal forces on the minimum geometrically unchangeable system under the balanced force system, and no internal forces and deformation occur in other parts of the frame or the truss.

(3) Load equivalent characteristic of the statically determinate structure

When the load acting on a geometrically unchangeable part of a statically determinate structure is equivalently transformed (the load distribution is different, but the resultant is the same), only the internal forces of that part change, and the internal forces of other parts do not change.

As shown in figure 3.49a, a concentrated load P acts on the midpoint of member AB of the statically determinate truss, and the axial forces of other members except AB are labeled as F_{N1} . Transform the load, and two loads of $P/2$ are applied on both ends A and B, as shown in figure 3.49b. The resultant is equal to P , which is the equivalent transformation of P in figure 3.49a. Herein, the axial forces of the rest members are labeled as F_{N2} . When two sets of loads are applied to the truss at the same time as shown in figure 3.49c, the axial forces of the rest members are $F_{N1} - F_{N2}$. These two sets of loads form a balanced force system. According to the local equilibrium characteristics of the statically determinate structure, the internal forces of the structural members outside the balance force system are zero, so the axial forces of the rest members are zero, which means $F_{N1} = F_{N2}$.

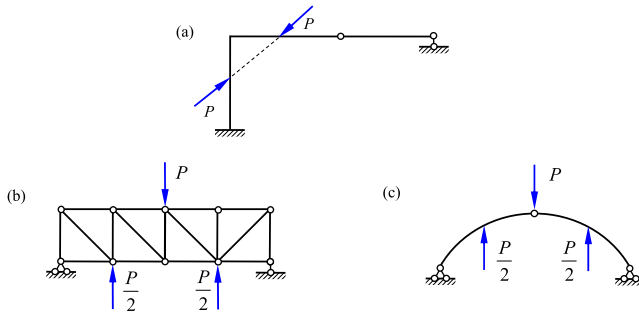


FIG. 3.48 – Local equilibrium characteristic of statically determinate structure.

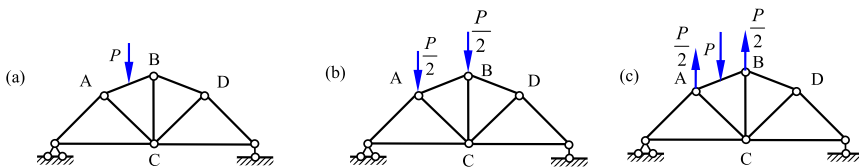


FIG. 3.49 – Load equivalent characteristic of statically determinate structure.

(4) Characteristic of geometrical construction transformation for the statically determinate structure

When a geometrically unchangeable part of the statically determinate structure undergoes a geometrical construction transformation, the internal forces of the rest parts remain unchanged.

As shown in figure 3.50a, the statically determinate truss is subjected to a load P , and member AB can be replaced with a small truss shown in figure 3.50b. Herein, only the internal forces of AB are changed, while the internal forces of the rest members of the truss are unchanged. Because the equilibrium of the rest parts can be maintained and the small truss can maintain equilibrium under the balanced force system composed of the original loads and constraint forces, the internal force states of the rest parts remain unchanged after the geometrical construction transformation.

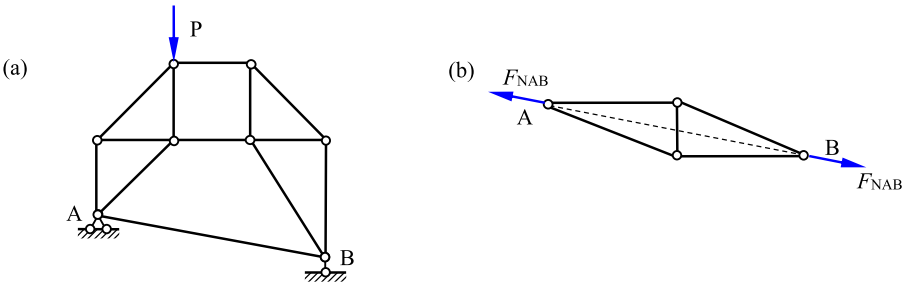


FIG. 3.50 – Construction transformation characteristic of statically determinate structure.

Problems

3.1 Use the method of sectional superposition to draw the moment diagrams of the following statically determinate beams.

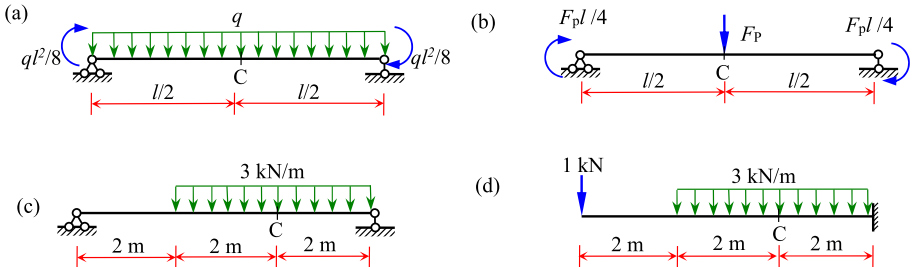


FIG. P3.1

3.2 Draw the internal force diagrams of the three inclined beams shown in the figure.

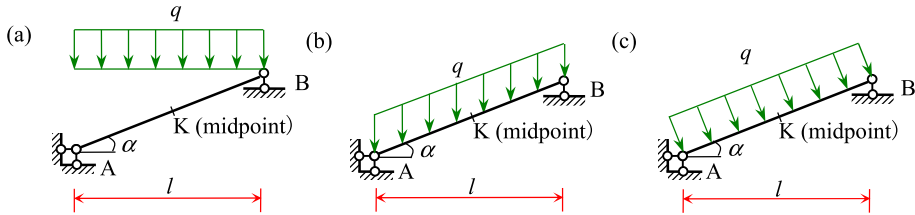


FIG. P3.2

3.3 Draw the moment and shear diagrams of the single-span beams shown in the figure.

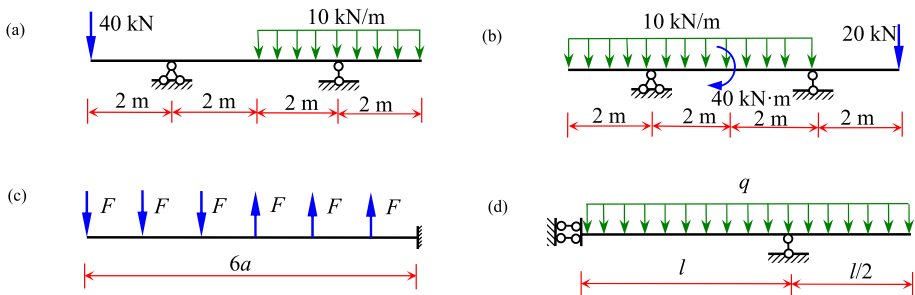


FIG. P3.3

3.4 Solve the support reactions of the multi-span beams shown in the figure, and draw the internal force diagrams.

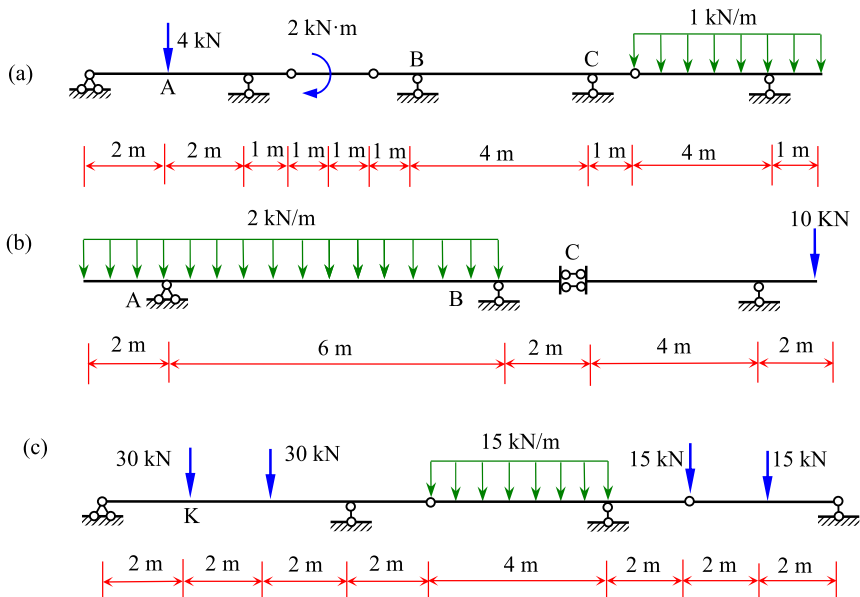


FIG. P3.4

3.5 Determine the position x of the hinge in the beam shown in the figure, so the absolute value of the moment at mid-span of the middle span and the moment of the support are equal.

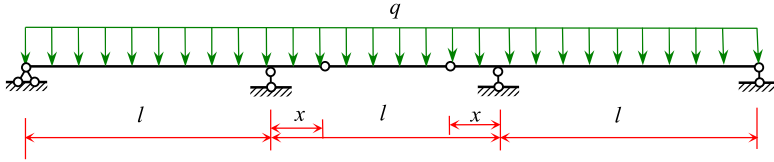


FIG. P3.5

3.6 Solve the internal forces of the specified members in the trusses shown in the figure.

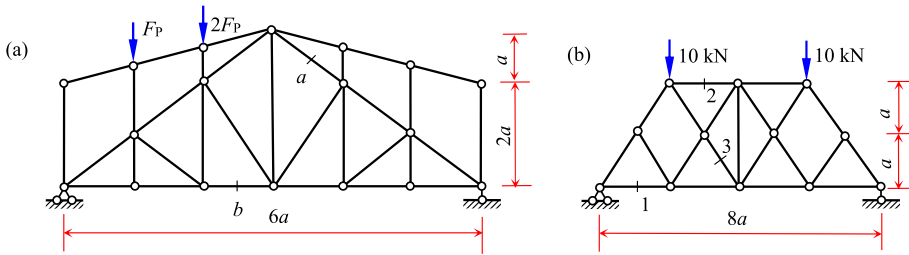


FIG. P3.6

3.7 Solve the internal forces of the members of the trusses shown in the figure.

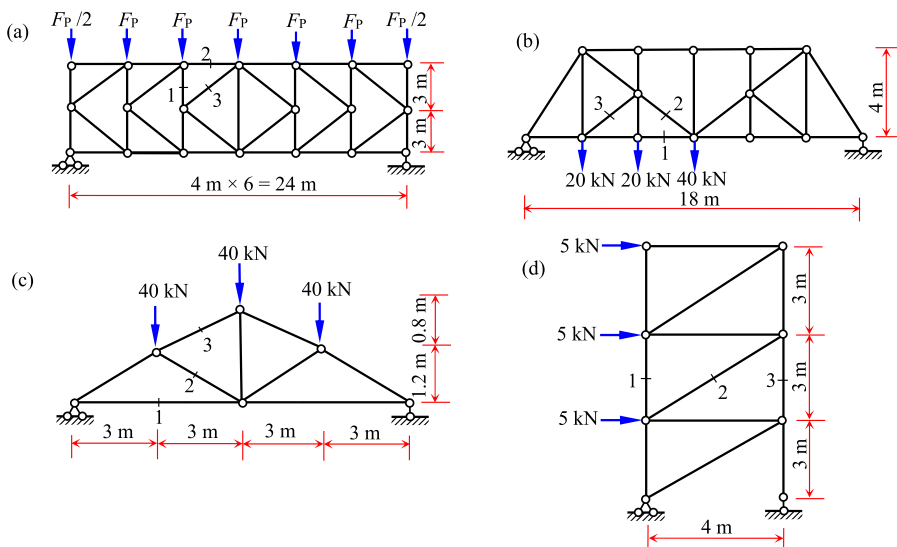


FIG. P3.7

3.8 Use the method of joints or method of sections to solve the internal forces of the members of each truss in the figure.

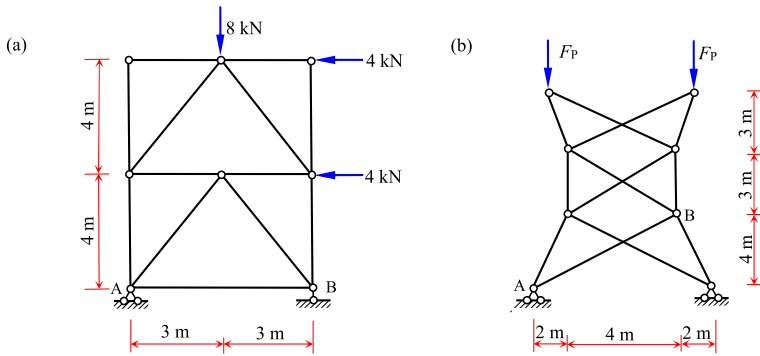


FIG. P3.8

3.9 Draw the moment diagrams quickly for the following frames.

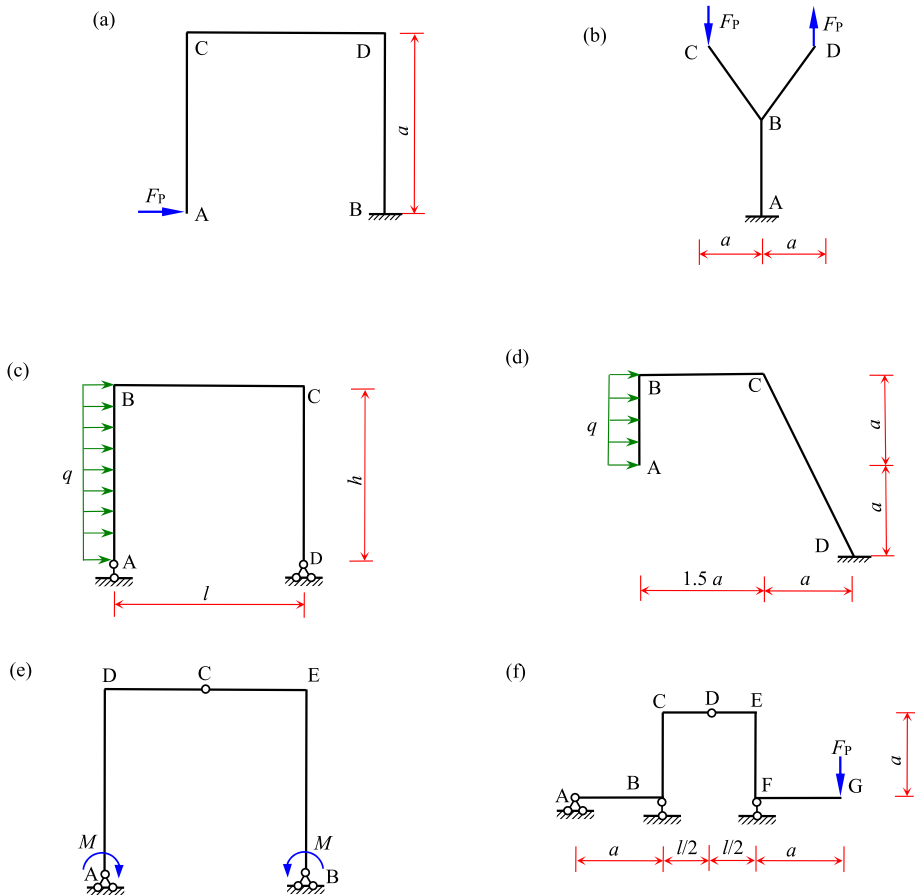


FIG. P3.9

3.10 Draw the moment diagrams for the following frames.

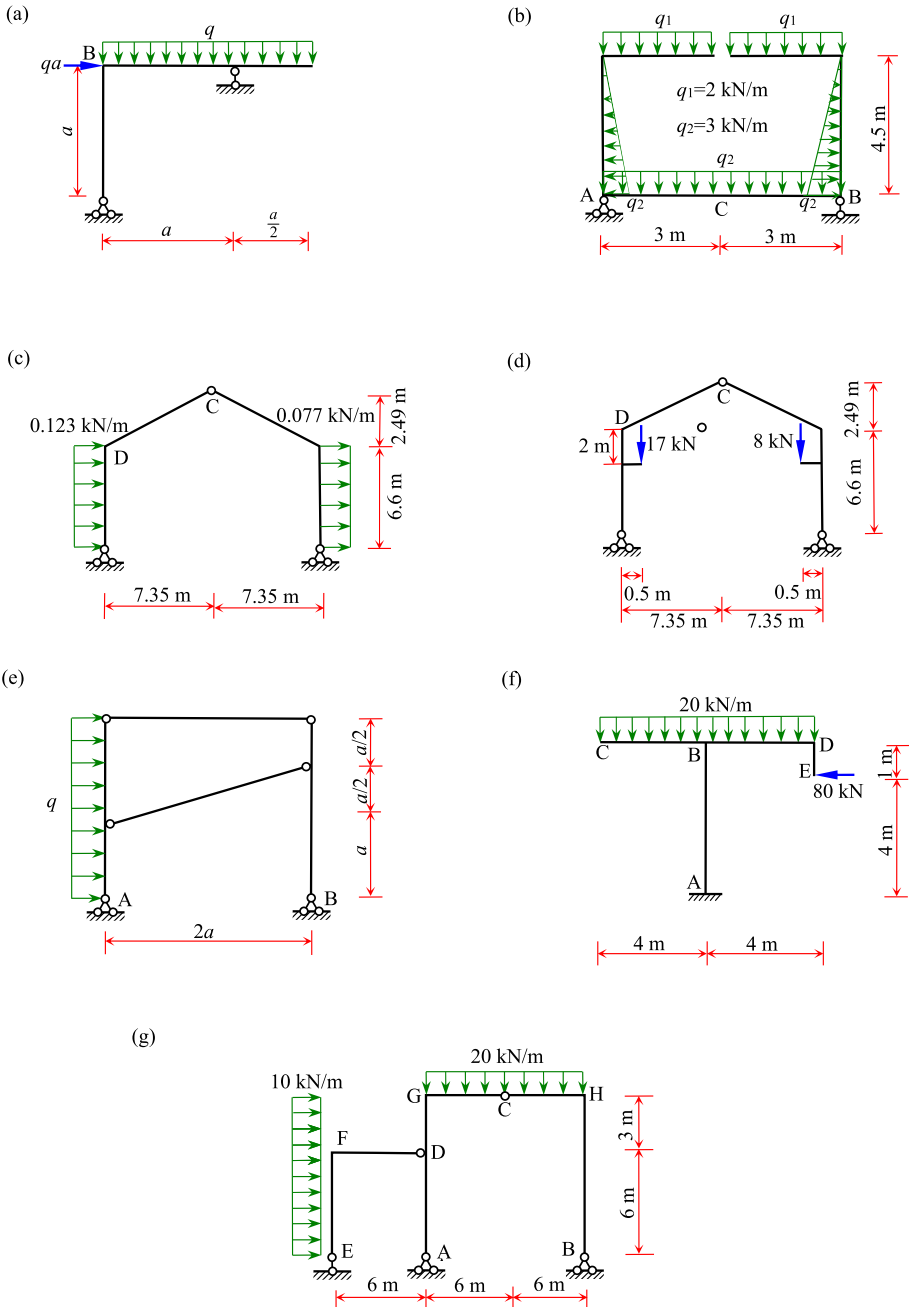


FIG. P3.10

3.11 Draw the internal force diagrams for the following composite structures.

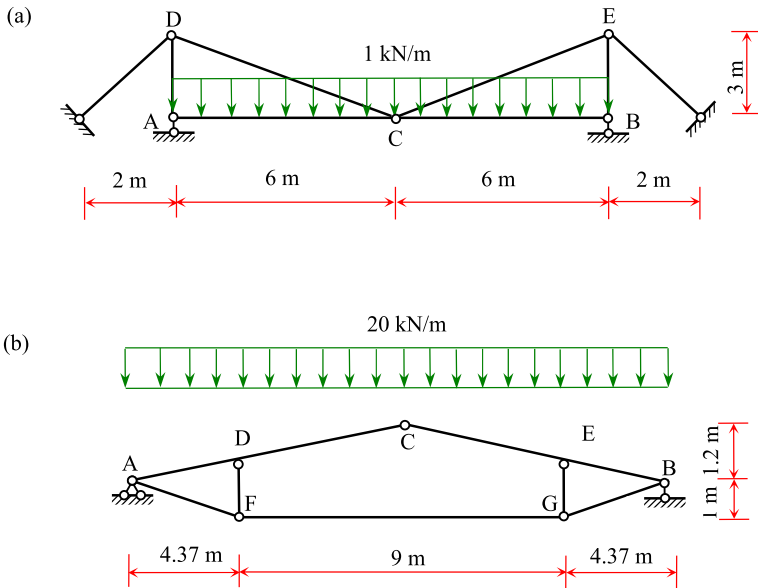


FIG. P3.11

3.12 The parabolic axis equation of the three-hinged arch shown in the figure is $y = \frac{4f}{l^2}x(l - x)$, $l = 16$ m, $f = 4$ m. Determine:

- The support reactions.
- The moment, shear, and axial force of section E.
- The shear and axial force of the left and right sections of point D.

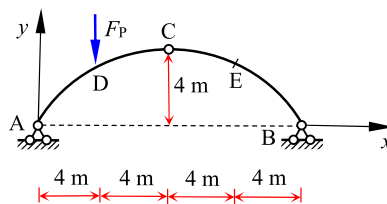


FIG. P3.12

3.13 Refer to the three-hinged arch in figure P3.12. Answer:

- If the arch height is changed ($f = 8$ m), how will the support reactions and moments change?
- If the arch height and span are changed at the same time, but the ratio of height to span remains unchanged, how will the support reactions and moments change?

Chapter 4

Principle of Virtual Work and Deflection Calculation

The two basic problems of structural analysis are internal force analysis and deflection calculation. This chapter mainly discusses the problem of deflection calculation of statically determinate structures. Firstly, the theoretical basis of the deflection calculation of member structures, *i.e.*, the principle of virtual work for deformable structures is discussed. The principle of virtual work comprises two forms: the principle of virtual forces and the principle of virtual displacements. The principle of virtual work can be used for resolving both the deflections and constraint forces, and it is also applicable for both statically determinate structures and indeterminate structures. It lays the theoretical foundation for solving statically indeterminate problems by energy method. Based on the principle of virtual work, the unit-load (or dummy load) method for structural deflection calculation and the general formula for deflection calculation of statically determinate structures under loads and temperature change, etc. can be derived. Then, the simple and practical graph multiplication method is introduced for fast computing of structural slopes and deflections. Finally, a brief introduction is given to the theorem of reciprocal works, the derived theorem of reciprocal displacements, the theorem of reciprocal reactions, and the theorem of reciprocal displacement and reaction.

4.1 Overview of Deflection Calculation

4.1.1 Concept of Structural Displacements

Engineering structures consist of deformable materials, which deform under loads or other external factors. The deformation causes position change in different places of structures, which is called the **displacement** of structures. For example, the statically determinate frame structure shown in figure 4.1 will undergo deformation and displacements as indicated by the dashed line under the given load.

The displacements of the structure can be divided into linear displacements and angular displacements. Linear displacement refers to the distance that a point on the structure moves relative to its original position. The relative position change between two points on the structure along the connecting line of the two points is called the relative linear displacement. The angular displacement refers to the rotation angle of a member section relative to its original position, and the relative rotation angle between two sections on the structure is referred to as the relative angular displacement. All these displacements can be collectively referred to as **generalized displacements**. For example, BB' and CC' in figure 4.1 represent the linear displacement of joint B and C respectively, θ_B denotes the angular displacement of rigid joint B, and $\Delta_{\theta C}$ indicates the relative angular displacement between the left and right sections of hinged joint C. Note that the portion at the right of joint C is subsidiary. When the load acts at the basic portion, the subsidiary portion has no internal force, thus only rigid body displacement occurs.

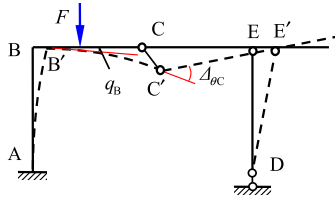


FIG. 4.1 – Deformation and displacements of a statically determinate frame under load.

The external factors that cause deflections of a structure mainly include:

(1) Loads

The structure deforms under loads, which will produce displacements.

(2) Temperature change

When the structure is subjected to temperature change, displacements occur due to the thermal expansion and contraction characteristics of the material.

(3) Support settlement

When the foundation base settles, the supports of the structure will move or rotate, causing displacements in the structure.

(4) Fabrication error

Dimension and shape errors may occur during the manufacturing process of structural members, causing displacements during the assembly of the structure.

In addition to the above factors, other factors such as shrinkage of material can also cause displacements in a structure.

4.1.2 Purpose of Deflection Calculation

(1) Check the stiffness requirements of structures

In addition to the bearing capacity or strength requirements stipulated by the design specifications, engineering structures should also meet the stiffness requirements. If the deformation of a structure under loads is too large, it cannot serve normally though it meets the strength requirements. For example, the ratio of maximum deflection to the structural span of bending members and beam-type structures should be less than the stipulated limit. This limit is various for different situations. For instance, the allowable deflection limit of the roof and floor beam is stipulated as $1/200$ – $1/400$ of the beam span, while that of the crane beam is usually taken as $1/600$ of the span.

As another example, for high-rise building structures, the ratio of inter-story drift to floor height and the ratio of top displacement to total height should be less than the stipulated limits under wind or earthquake loads. These regulations guarantee the comfort of living in high-rise buildings, the normal opening of doors and windows, no crack and damage in decorations, the safety of pipelines, and the normal operation of elevators, etc. Sometimes, to meet the designed structural configuration requirements, it is necessary to pre-calculate and take into account the displacements. For example, in long-span beams and trusses, it is often necessary to make pre-arch during the production, so as to avoid obvious downward deflection in service.

(2) Analyze statically indeterminate structures

For statically indeterminate structures, the internal forces cannot be uniquely determined from the static equilibrium conditions. Compatibility requirements must also be considered in the solution, which requires calculating the deflections of structures. For example, the beam shown in figure 4.2a has a redundant constraint, which is statically indeterminate. The four support reactions shown in the figure cannot be completely determined from the three equilibrium equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. If the deflection calculation problem of the structure is solved, the statically determinate beam in figure 4.2b can be utilized as the computational model. According to the compatibility requirement that the total vertical displacement of point B should be equal to zero under the combined action of external load q and reaction F_{yB} , the value of F_{yB} can be determined, and then the internal forces of the beam can be solved.

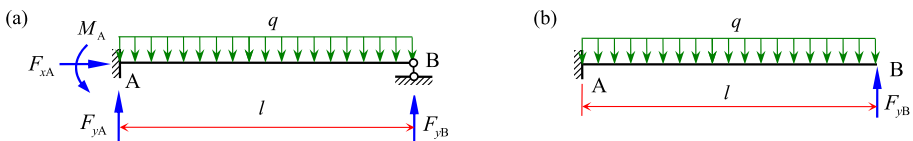


FIG. 4.2 – (a) A statically indeterminate beam, (b) the corresponding statically determinate beam.

4.2 Principle of Virtual Work for Deformable Structures

The **principle of virtual work** is a fundamental principle in mechanics. Different virtual objects, it has two forms: the **principle of virtual displacements** and the **principle of virtual forces**. These principles have many aspects of applications in structural analysis. The equilibrium condition of force can be derived from the principle of virtual displacements of deformable structures, while the compatibility requirements of displacement can be deduced from the principle of virtual forces. By incorporating the constitutive relation of elastic materials, a series of energy principles and energy methods can be derived from the principle of virtual displacements and the principle of virtual forces, which can be used to analyze various mechanical problems of elastic structures.

4.2.1 Principle of Virtual Work for Rigid Body System

For rigid body systems with idealized constraints, the principle of virtual work can be expressed as follows:

Assuming that there is an arbitrary equilibrium force system acting on a rigid body system, and the system has any infinitesimal rigid body displacements that meet the constraint conditions, then the sum of virtual work done by the active forces (external forces) undergoing the virtual displacements is zero, *i.e.*

$$W = 0 \quad (4.1)$$

Here are two states that are independent of each other: one is the arbitrary equilibrium force system acting on the rigid body system, referred to as the equilibrium force system; another is the arbitrary infinitesimal rigid body displacements of the system that meet the constraint conditions, referred to as the possible displacements. The so-called “virtual work” means that both sides of doing work, *i.e.*, equilibrium force system and possible displacements, are independent of each other. In the principle of virtual work, the equilibrium force system is independent of the possible displacements. Therefore, either the displacements or the force system can be regarded as virtual. Hence, the principle has two forms: the principle of virtual displacements and the principle of virtual forces.

For ideal constraints, such as smooth hinges and rigid links, the work of the constraint forces undergoing possible displacements constantly equals zero. Since the distance between any two points on a rigid body remains unchanged, any two points can be considered to be connected by a rigid link. Therefore, the rigid body can be regarded as a system of particles with ideal constraints, and the work done by the internal forces undergoing the possible displacements constantly equals zero.

4.2.2 Application Conditions of Principle of Virtual Work for Deformable Structures

In the principle of virtual work for rigid body systems, as the work done by internal forces is constantly zero, it is only necessary to consider the work done by external

forces. While for deformable systems, due to the existence of strain in the deformable structure, the principle of virtual work should take into account the work done by both external forces and internal forces. In other words, it is imperative to supplement the internal virtual work done by the stress on the deformation. This is an important difference between the principle of virtual work for deformable structures and rigid body systems.

The **principle of virtual work for deformable structures** can be expressed as follows: assuming a deformable structure is in an equilibrium state under a force system, and there is tiny continuous deformation in the deformable structure caused by other factors that meet the constraint conditions, then the external virtual work done by external forces on the displacements, W_e , constantly equals to the internal virtual work done by the stress resultant on the deformation of all differential segments, W_i (also called the deformation virtual work, or virtual strain energy).

Therefore, the principle of virtual work for deformable structures can be expressed in the form of a virtual work equation:

$$W_e = W_i \quad (4.2)$$

The application of the principle of virtual work for deformable structures needs to meet the following two conditions: the force system should satisfy the equilibrium conditions; the displacements should conform to the support conditions and maintain the continuity of the structure, that is, the displacements should satisfy the compatibility conditions.

The two conditions are further explained as follows:

(1) Equilibrium conditions

Take the beam AB subjected to the axial load (the load intensity is p) and normal load (the intensity is q) along the length of the beam as shown in figure 4.3a for example. The external forces at point A are M_A , F_{NA} , F_{SA} , and the external forces at point B are M_B , F_{NB} , and F_{SB} . If the beam is in an equilibrium state, the differential element in figure 4.3b should meet the equilibrium conditions, that is, the following equilibrium differential equations between the internal forces M , F_N , F_S , and the distributed loads p and q should be satisfied.

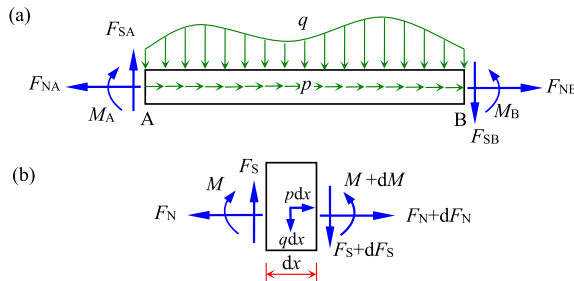


FIG. 4.3 – (a) A beam under axial and normal distributed loads, (b) equilibrium conditions of differential element.

$$\left. \begin{aligned} dF_N + p dx &= 0 \\ dF_S + q dx &= 0 \\ dM - F_S dx &= 0 \end{aligned} \right\} \quad (4.3)$$

(2) Compatibility conditions

The displacements and deformation of beam AB are illustrated in figure 4.4a. The displacements of any section can be described by three components, namely the axial displacement u and transverse displacement w of the section centroid, and the angular displacement θ of the section. The angular displacement φ of the tangent of the beam axis can be derived from w :

$$\varphi = \frac{dw}{dx} \quad (4.4)$$

The deformation of the differential element dx can be described by three strain components, namely the linear strain ε of the axis, the average shear strain γ_0 of the section, and the relative rotation angle $d\theta$ between the cross-section at both ends in figure 4.4b.

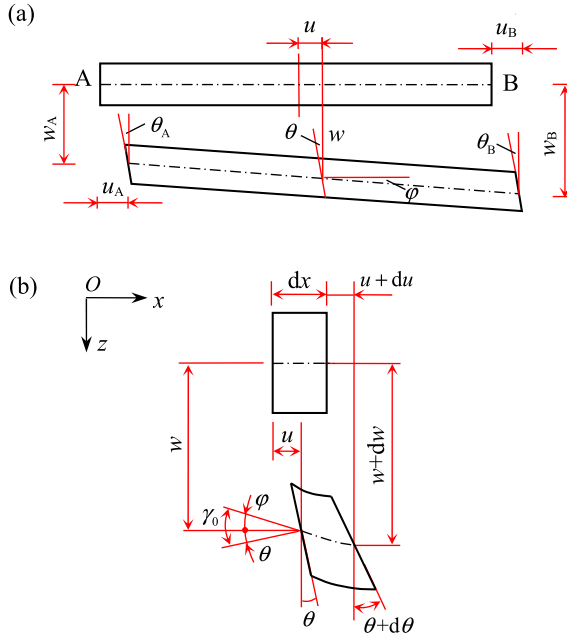


FIG. 4.4 – (a) Displacements and deformation of the beam, (b) strain and displacement components of differential element.

The following relations between the strain components and displacement components should be satisfied:

$$\varepsilon = \frac{du}{dx} \quad (4.5)$$

$$\gamma_0 = \theta + \varphi = \theta + \frac{dw}{dx} \quad (4.6)$$

Besides, the static equilibrium or geometric boundary conditions should be met at both ends of the beam. Taking section A as an example, if it is a fixed end, the u , w , and θ of section A should be equal to the displacements u_A , w_A , and θ_A of the support; if it is a hinged end, the u , w and M of section A should be equal to u_A , w_A , and M_A of the support; if it is a free end, the F_N , F_S , and M of section A should be equal to the external forces F_{NA} , F_{SA} , and M_A given at this section. The above-mentioned strain-displacement relations and geometric boundary conditions are collectively referred to as deformation compatibility conditions (or consistent deformation conditions). Note that, the sectional deformations of the beam contain the shear deformation discussed herein, and such a beam is just the Timoshenko beam.

4.2.3 Virtual Work Equation for Deformable Structures

To derive the virtual work equation of deformable structures, let the equilibrium force state in figure 4.3 do virtual work on the continuous deformation state shown in figure 4.4, then the external virtual work done by external forces on the displacements is

$$W_e = (M_B \theta_B + F_{NB} u_B + F_{SB} w_B) - (M_A \theta_A + F_{NA} u_A + F_{SA} w_A) + \int_A^B (pu + qw) dx \quad (4.7)$$

where, the first two items at the right side of the equation indicate the virtual work done by member end forces, and the third item is the virtual work done by the distributed loads.

The internal virtual work done by the stress resultant undergoing the deformation of the differential element dx is:

$$dW_i = Md\theta + F_N \varepsilon dx + F_S \gamma_0 dx$$

Integrating the above formula over the entire length of beam AB, we can obtain the internal virtual work of the entire deformable body:

$$W_i = \int_A^B (Md\theta + F_N \varepsilon dx + F_S \gamma_0 dx) \quad (4.8)$$

Substituting equations (4.7) and (4.8) into equation (4.2), yields the virtual work equation for deformable structures:

$$\begin{aligned}
 & (M_B \theta_B + F_{NB} u_B + F_{SB} w_B) - (M_A \theta_A + F_{NA} u_A + F_{SA} w_A) + \int_A^B (pu + qw) dx \\
 & = \int_A^B (M d\theta + F_N \varepsilon dx + F_S \gamma_0 dx)
 \end{aligned} \tag{4.9}$$

The virtual work equation (4.9) for deformable structures can be proved as follows.

Firstly, according to the equilibrium differential equation (4.3), the following equation can be derived:

$$\int_A^B [(dF_N + p dx)u + (dF_S + q dx)w + (dM - F_S dx)\theta] = 0$$

Rewrite the above equation as

$$\int_A^B (u dF_N + w dF_S + \theta dM) + \int_A^B (pu + qw - F_S \theta) dx = 0 \tag{a}$$

Since

$$u dF_N + w dF_S + \theta dM = d(uF_N + wF_S + M\theta) - (F_N du + F_S dw + M d\theta)$$

Equation (a) can be rewritten as

$$(uF_N + wF_S + M\theta)|_A^B - \int_A^B (F_N du + F_S dw + M d\theta) + \int_A^B (pu + qw - F_S \theta) dx = 0 \tag{b}$$

From the strain-displacement relations (4.5) and (4.6), we have

$$du = \varepsilon dx, \quad dw + \theta dx = \gamma_0 dx$$

Substituting them into equation (b) yields:

$$\begin{aligned}
 & (F_{NB} u_B + F_{SB} w_B + M_B \theta_B) - (F_{NA} u_A + F_{SA} w_A + M_A \theta_A) + \int_A^B (pu + qw) dx \\
 & = \int_A^B (F_N \varepsilon dx + F_S \gamma_0 dx + M d\theta)
 \end{aligned}$$

This is just equation (4.9). Thus, the virtual-work equation for deformable structures is proved.

If the beam is also subjected to concentrated loads F_P in addition to the distributed loads, it is only necessary to add the virtual work done by the concentrated loads $\sum F_P \Delta$ to the external virtual work W_e (Δ is the displacement corresponding to F_P), then we can obtain the generalized virtual-work equation for deformable structures. In fact, by incorporating the Dirac δ function, a discrete concentrated load can be regarded as a continuous distributed load, thus the virtual work done by the concentrated loads can be computed as $\sum F_P \Delta$.

In the virtual work equation (4.9) of deformable structures, only a single beam is discussed. For a member structure, it is only necessary to apply equation (4.9) to each member in the structure, and then superimpose them to yield

$$\begin{aligned}
& \sum (M\theta + F_N u + F_S w)|_A^B + \sum \int_A^B (pu + qw)ds + \sum F_P \Delta \\
& = \sum \int_A^B (M d\theta + F_N \varepsilon ds + F_S \gamma_0 ds)
\end{aligned} \quad (4.10)$$

in which, the left three items mean the sum of the virtual work done by the member end forces, the distributed loads, and the concentrated loads of all the members, and the right-side term is the sum of the virtual work done by the internal forces on the deformation of all the members.

According to different load states, the member end sections of all members can be divided into two categories: the first type is the member end sections at the inner joints of the structure (as shown in figure 4.5, sections B1, B2 and B3 of members 1, 2, 3). This kind of joint is in an equilibrium state by itself, that is, the end forces of all members around the joint form an equilibrium force system, so the sum of virtual work done by the forces on the joint displacements is equal to zero.

The second type is the boundary sections of the structure (such as sections A, C, and D in figure 4.5). The sum of virtual work done by these member end forces is just the virtual work done by the external forces on the structure boundaries, including virtual work done by the boundary loads and support reactions. Among them, the virtual work of support reactions is written as $\sum F_{RK} c_K$, where F_{RK} represents the support reaction, and c_K is the displacement of the support corresponding to F_{RK} .

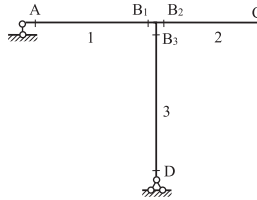


FIG. 4.5 – The classification of member end sections.

Generally, the virtual work done by the structural boundary loads and that by the concentrated loads of each member can be expressed as a unified form $\sum F_P \Delta$, so the general form of the virtual-work equation for the member structure can be written as:

$$\sum F_P \Delta + \sum \int (pu + qw)ds + \sum F_{RK} c_K = \sum \int (M d\theta + F_N \varepsilon ds + F_S \gamma_0 ds) \quad (4.11)$$

where, the first and second items at the left side represent the virtual work done by the concentrated loads and distributed loads, respectively, which can be combined into one item and still written as $\sum F_P \Delta$. Then, equation (4.11) can be further simplified as (let $d\theta = \kappa ds$, κ is the curvature of the axis)

$$\sum F_P \Delta + \sum F_{RK} c_K = \sum \int (M \kappa + F_N \varepsilon + F_S \gamma_0) ds \quad (4.12)$$

To understand and apply the principle of virtual work for deformable structures, the following basic concepts should be noted:

- (1) The equilibrium state of the force system and the compatibility state of displacements involved in the principle of virtual work are independent of each other, without causal relation. That is, the displacements are not caused by the internal or external forces of the original equilibrium state but are the possible displacements caused by any other factors. Consequently, the work is called virtual work.
- (2) The possible displacements must be arbitrary and infinitesimal. If the possible displacements are taken as finite quantities, the principle of virtual work will be invalid. In other words, it can only be regarded as an approximate application of the principle of virtual work under this situation.
- (3) The expression of the principle of virtual work for deformable structures does not involve the type of deformable structures, the property of materials, and the magnitude of deformation or displacements. Although the small deformation hypothesis is adopted for the sake of simplification in the above-proving procedure, the principle of virtual work is suitable to any type of structure and is applicable to material nonlinear and geometric nonlinear problems.

4.2.4 Principle of Virtual Forces and Principle of Virtual Displacements

The principle of virtual work involves two independent states, *i.e.*, equilibrium force system and possible displacements. According to different needs, either the force system or the displacements can be selected as virtual. By selecting different virtual objects, the principle of virtual work can be transformed into two forms: the principle of virtual forces and principle of virtual displacements. The derived virtual-force equation and the virtual-displacement equation can become the necessary and sufficient conditions for the compatibility equations and equilibrium equations of the force system, respectively. The principle of virtual forces and the principle of virtual displacements can be used to solve two kinds of problems: taking virtual force system to solve the unknown displacements, such as the unit-load method for computing structural displacements; taking virtual displacements to solve the unknown forces, such as the unit support displacement method for computing the reactions, and the kinematic method for constructing influence lines of internal forces of structures under moving loads.

(1) Principle of virtual forces and virtual-force equation

Take virtual equilibrium force system, and examine whether the deformation state satisfies the compatibility equations. Through the requirement that the external virtual work equals the internal virtual work, the virtual-work equation can be written as

$$\sum \bar{F}_P \Delta + \sum \bar{F}_{RK} c_K = \sum \int (\bar{M} \kappa + \bar{F}_N \varepsilon + \bar{F}_S \gamma_0) ds \quad (4.13)$$

where, $\bar{F}_P$, $\bar{F}_{RK}$, $\bar{M}$, $\bar{F}_N$ and $\bar{F}_S$ form an arbitrary virtual equilibrium force system, Δ , c_K , κ , ε and γ_0 are the physical quantities of the deformation state to be investigated. Equation (4.13) is called the virtual-force equation for deformable structures.

The principle of virtual forces for deformable structures can be described as follows. Under the premise that the virtual force system is arbitrary and satisfies the equilibrium equations, if the virtual-force equation (4.13) is true, then the deformation state to be examined must satisfy the compatibility equations. On the contrary, under the same premise, if the deformation state is known to satisfy the compatibility equations, then the virtual-force equation (4.13) must be true. In summary, under the above-mentioned premise, the virtual-force equation (4.13) is a necessary and sufficient condition for the compatibility equations. In other words, equation (4.13) and the compatibility equations are equivalent; that is, equation (4.13) is a compatibility equation expressed by the form of virtual work.

(2) Principle of virtual displacements and virtual-displacement equation

Take virtual compatible deformations, and examine whether the force system satisfies the equilibrium equations. Through the requirement that the external virtual work equals the internal virtual work, the virtual-work equation can be formulated as

$$\sum F_P \bar{\Delta} + \sum F_{RK} \bar{c}_K = \sum \int (M \bar{\kappa} + F_N \bar{\varepsilon} + F_S \bar{\gamma}_0) ds \quad (4.14)$$

where, $\bar{\Delta}$, $\bar{c}_K$, $\bar{\kappa}$, $\bar{\varepsilon}$ and $\bar{\gamma}_0$ form an arbitrary virtual compatible deformation state, F_P , F_{RK} , M , F_N and F_S are the physical quantities of the force system to be investigated. Equation (4.14) is named the virtual-displacement equation for deformable structures.

The principle of virtual displacements for deformable structures can be described as follows. Under the premise that the virtual deformation state is arbitrary and satisfies the compatibility equations, if the virtual-displacement equation (4.14) is true, then the force system to be examined must satisfy the equilibrium equations. On the contrary, under the same premise, if the force system is known to satisfy the equilibrium equations, then the virtual-displacement equation (4.14) must be true. In summary, under the above-mentioned premise, the virtual-displacement equation (4.14) is a necessary and sufficient condition for the equilibrium equations of the force system. In other words, equation (4.14) and the equilibrium equations of the force system are equivalent; that is, equation (4.14) is a force system equilibrium equation expressed by the form of virtual work.

In the mechanics of deformable structures, the deflections, and forces are a pair of dual quantities. The principle of virtual work reflects the dual-complementary relation between the displacements and forces. The thought of duality and symmetry plays an important role in the development of theoretical methods of mechanics, mathematics and physics. For proof of the principle of virtual forces and

the principle of virtual displacements for deformable structures, the readers can refer to related textbooks.

4.3 Unit-Load Method for Structural Deflection Calculation

In this section, we will utilize the virtual-force equation of plane member structures to derive the general formula of structural deflection calculation and establish the unit-load method for solving the deflections of specified sections of structures.

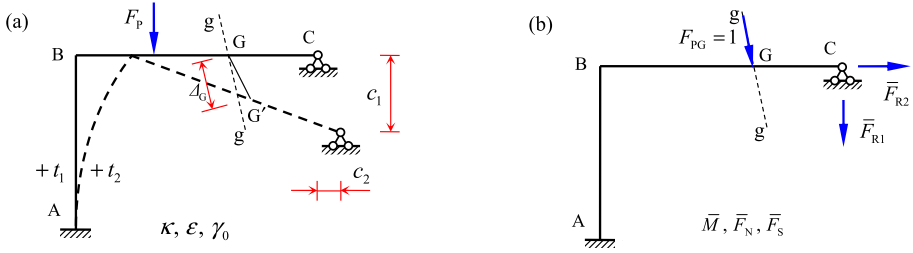


FIG. 4.6 – (a) Actual deformation state of a frame under load, support settlement and temperature change, (b) virtual equilibrium force system of the frame.

As shown in figure 4.6a, the actual deformation state of a frame under the load, support settlement, and temperature change is illustrated by the dashed line. The point G on the structure moves to an unknown position G' after deformation. To determine the displacement Δ_G of point G along any specified direction g-g in the actual deformation state, we can take a virtual equilibrium force system as shown in figure 4.6b, where, a unit load $F_{PG} = 1$ is applied at point G along the g-g direction of the undetermined displacement Δ_G . Denote the resulting internal forces of the structure as $\bar{M}$, $\bar{F}_N$ and $\bar{F}_S$, and the reaction as $\bar{F}_R$.

The external virtual work done by the external forces of the virtual force system on the corresponding displacements of the actual deformation state can be written as

$$W_e = F_{PG}\Delta_G + \bar{F}_{R1}c_1 + \bar{F}_{R2}c_2 = 1 \times \Delta_G + \sum \bar{F}_{RK}c_K$$

The internal virtual work done by the stress resultant (internal force) of the virtual force system on the corresponding strain of the actual deformation state is

$$W_i = \sum \int (\bar{M}\kappa + \bar{F}_N\varepsilon + \bar{F}_S\gamma_0)ds$$

According to the virtual force equation for deformable structures, $W_e = W_i$, we have

$$\Delta_G = \sum \int (\bar{M}\kappa + \bar{F}_N\varepsilon + \bar{F}_S\gamma_0)ds - \sum \bar{F}_{RK}c_K \quad (4.15)$$

where, κ , ε and γ_0 are respectively the curvature, axial strain, and average shear strain of the members in actual condition.

Equation (4.15) is a general formula for computing the displacements of member structures, which is applicable to both statically determinate structures and statically indeterminate structures. Therefore, as long as the internal forces $\bar{M}$, $\bar{F}_N$, $\bar{F}_S$ and reaction $\bar{F}_{RK}$ of the virtual force system are solved, and the support settlement, the axial strain, average shear strain, and curvature of the members at the actual deformation state are all known, the displacement Δ_G can be calculated by equation (4.15). If the result is positive, which means the virtual work done by the unit load is positive, then the actual direction of the undetermined displacement Δ_G is the same as the virtual unit load $F_{PG} = 1$. If the result is negative, then the direction of displacement is opposite.

It can be seen from the above analysis that, to determine the displacement of the specified section of a structure by the principle of virtual force, the key is to set a proper equilibrium force system. The cleverness of this method lies in that a unit load is set at the position of undetermined displacement along its direction so that the virtual work just equals the undetermined displacement numerically. This method of calculating structural displacements based on the principle of virtual work is called the **unit-load method**, which is an energy method.

To facilitate the setting of virtual unit load, the concept of generalized force corresponding to the generalized displacement can be introduced. Linear displacements, angular displacements, relative linear displacements, relative angular displacements, and a set of displacements can be collectively referred to as generalized displacements. Concentrated forces, couples, a pair of concentrated forces, a pair of couples and a force system can be collectively referred to as **generalized forces**. To calculate any generalized displacement, the applied load of the virtual force system should be a unit generalized force corresponding to the generalized displacement. Here, “corresponding”, also called “conjugate”, refers to the corresponding relationship in the work ($W = F\Delta$) between the force F and the displacement Δ . For example, a concentrated force is corresponding to a linear displacement, and a couple is conjugate to an angular displacement, etc. In this case, a force is a conjugate force of a displacement, and *vice versa*. The force and displacement are a pair of dual quantities.

In practical application, in addition to calculating the linear displacements of structures, sometimes it is necessary to calculate the generalized displacements, such as angular displacements and relative displacements. If an absolute linear displacement is to be calculated, a corresponding unit concentrated force should be applied at the position of the linear displacement along its direction. If an absolute angular displacement is to be calculated, a corresponding unit concentrated couple should be applied at the position of the angular displacement along its direction. If a relative displacement is to be calculated, a pair of reverse unit forces or couples

(generalized forces) should be applied along the direction of the relative displacement. It is noted that the dimension of the unit load is 1.

Taking the frame shown in figure 4.7a as an example, if the vertical or horizontal displacement component of point E is to be calculated, the unit load of the virtual force system should be exerted as shown in figure 4.7b or c, namely, a unit load is applied at point E along the vertical or horizontal direction. If the rotation angle of section E is to be calculated, a unit couple should be applied at point E in the virtual state as shown in figure 4.7d. If the relative linear displacement between points F and G, and the relative angular displacement between sections F and G are to be computed, then the unit loads of the virtual state are displayed in figure 4.7e, f, separately.

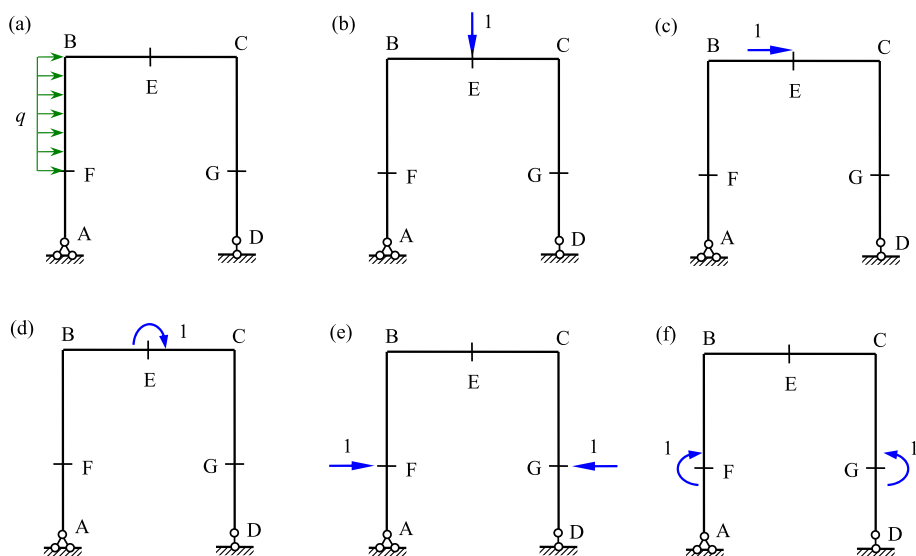


FIG. 4.7 – The unit loads corresponding to different displacements of a frame.

To calculate the linear displacement of a truss joint, the virtual unit load can also be applied in the above way. Taking the truss shown in figure 4.8a as an example, if the vertical displacement of joint C or the distance change between joints E and F is to be calculated, the unit force shown in figure 4.8b or c should be applied respectively. If the angular displacement of member CF is to be calculated, since the member can only bear axial force, the virtual unit couple should be a pair of reverse concentrated forces applied at both ends of the member, with the force direction perpendicular to the member and the magnitude equal to the reciprocal of the member length, as demonstrated in figure 4.8d. The virtual unit generalized force exhibited in figure 4.8e can be used to calculate the relative rotation angle between members CF and CG.

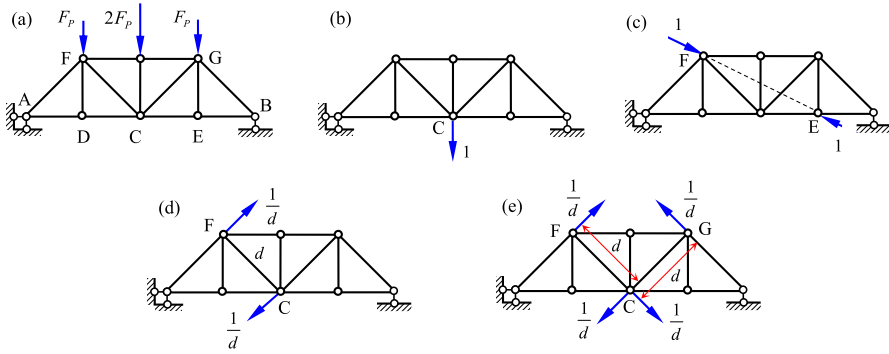


FIG. 4.8 – The unit loads corresponding to different displacements of a truss.

4.4 Deflection Calculation under Loads

In this section, the calculation of deflections at specific sections of linear elastic plane member structures under loads will be discussed, focusing on the analytical integral method of statically determinate structures.

4.4.1 Formula for Deflection Calculation under Loads

When only loads are applied, the general formula (4.15) for calculating structural deflections can be simplified as

$$\Delta = \sum \int (\bar{M}\kappa + \bar{F}_N\varepsilon + \bar{F}_S\gamma_0)ds \quad (4.16)$$

where, $\bar{M}$, $\bar{F}_N$ and $\bar{F}_S$ indicate structural internal forces caused by the unit load of the virtual force system, κ , ε and γ_0 are the curvature, axial strain, and average shear strain of the infinitesimal element of members caused by the loads in the actual deformation state, *i.e.*, the bending, tensile and shear strain.

The moment, axial force, and shear force of the bending members in the actual deformation state are denoted as M_P , F_{NP} and F_{SP} , respectively. Here, the internal forces are caused by loads, so they are indicated by the subscript P. The internal forces will cause stresses, and then yield strains. According to the theory of mechanics of materials, the strain components of the section in linear elastic members can be written as

$$\kappa = \frac{M_P}{EI}, \quad \varepsilon = \frac{F_{NP}}{EA}, \quad \gamma_0 = k \frac{F_{SP}}{GA} \quad (4.17)$$

where, E and G denote the elastic modulus and shear modulus of the material respectively; I and A are the moment of inertia and area of the member section separately; EI , EA , and GA represent the bending, tensile and shear stiffness of the member section, respectively; k indicates a coefficient related to the shape of the

section introduced by considering the uneven distribution of shear stress along the section, which is called the sectional shape coefficient and expressed as:

$$k = \frac{A}{I^2} \int_A \frac{S^2}{b^2} dA \quad (4.18)$$

in which, b means the width of the section where the shear stress is taken, and S is the static moment of the area below (or above) the point where the shear stress is taken with respect to the neutral axis of the section. The values of k for several common cross-sections are listed in table 4.1.

TAB. 4.1 – Sectional shape coefficients k of shear strain.

Section form	Coefficient k
Rectangle	6/5
Circle	10/9
Thin-wall ring	2
I-shaped or box-shaped	A/A_1 (A_1 is the area of the web)*

*This is an approximate value.

Substituting equation (4.17) into equation (4.16), the general formula for calculating the deflection of plane member structures under loads is obtained:

$$\Delta = \sum \int \frac{\overline{M}M_P}{EI} ds + \sum \int \frac{\overline{F}_N F_{NP}}{EA} ds + \sum \int \frac{k \overline{F}_S F_{SP}}{GA} ds \quad (4.19)$$

In equation (4.19), there are two kinds of internal forces:

M_P , F_{NP} , and F_{SP} — bending moment, axial force, and shear force caused by actual loads;

$\overline{M}$, $\overline{F}_N$, and $\overline{F}_S$ — bending moment, axial force, and shear force caused by the virtual unit load.

The sign convention of internal forces is summarized as follows:

Bending moments M_P and $\overline{M}$ — only the sign of the product $\overline{M}M_P$ is specified. When $\overline{M}$ and M_P cause tension on the same side of the member, the product takes a positive value;

Axial forces F_{NP} and $\overline{F}_N$ — tensile forces are positive;

Shear forces F_{SP} and $\overline{F}_S$ — those cause the differential element to rotate clockwise are positive.

4.4.2 Deflection Formulas for Various Structures

Equation (4.19) is a general formula for calculating the elastic deflection of plane member structures under loads, which is applicable to both statically determinate and indeterminate structures. The first, second, and third items on the right side of the equation represent the effects of bending, stretching, and shear deformation,

respectively. In fact, different structural forms have different stress and deformation characteristics, and the above three kinds of deformation have different contributions to the displacements. In general, the items that contribute little to displacements are often ignored for different structures, which not only meets the requirement of engineering accuracy, but also simplifies the calculation.

(1) Beams and frames

For beams and frames, the displacement is mainly caused by the bending deformation of the members, while the effects of tensile and shear deformation can be omitted generally. Thus, the displacement formula can be simplified as:

$$\Delta = \sum \int \frac{\bar{M} M_P}{EI} ds \quad (4.20)$$

(2) Trusses

For trusses, the members are only subjected to axial force, and the cross-sectional area A , axial forces F_{NP} and $\bar{F}_N$ are generally constant along the length of the members, so the displacement formula can be simplified as:

$$\Delta = \sum \int \frac{\bar{F}_N F_{NP}}{EA} ds = \sum \frac{\bar{F}_N F_{NP} l}{EA} \quad (4.21)$$

(3) Composite structures

For truss-beam composite structures, the influence of bending deformation should be considered for beams, while only the influence of tensile deformation needs to be considered for trusses. Accordingly, the displacement formula can be simplified as:

$$\Delta = \sum \int \frac{\bar{M} M_P}{EI} ds + \sum \int \frac{\bar{F}_N F_{NP}}{EA} ds \quad (4.22)$$

(4) Arches

For arches, when ignoring the influence of arch axis curvature, the displacement can still be calculated approximately by equation (4.19). Generally, it is only necessary to consider the effects of bending deformation. However, when the arch axis is close to the pressure line (*i.e.*, the distance between them is approximately the height of the arch section), or when calculating the horizontal displacement of flat arches ($f/l < 1/5$), the influence of tensile deformation should also be considered, *i.e.*,

$$\Delta = \sum \int \frac{\bar{M} M_P}{EI} ds + \sum \int \frac{\bar{F}_N F_{NP}}{EA} ds \quad (4.23)$$

4.4.3 Examples of Deflection Calculation under Loads

Example 4.1. Determine the vertical displacement Δ of point A on the cantilever beam shown in figure 4.9a and compare the effects of bending deformation and shear deformation on the displacement. The cross-section of the beam is a rectangle.

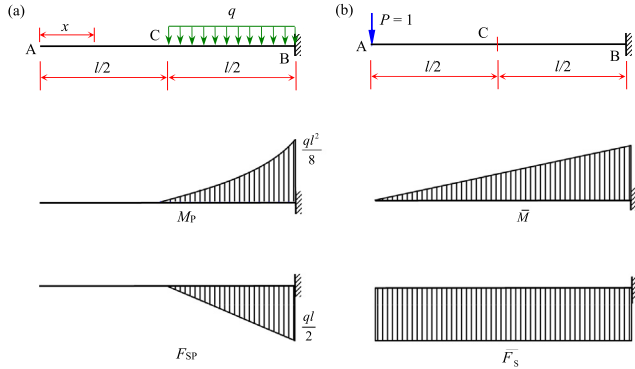


FIG. 4.9 – Deflection calculation of a cantilever beam under load.

Solution: To calculate the displacement, the internal forces under the actual load (figure 4.9a) and the virtual unit load (figure 4.9b) should be determined first. Taking point A as the coordinate origin, the internal forces of any section x are calculated as follows:

	Actual load	Virtual unit load
	$M_P = 0$	$\bar{M} = -x$
Segment AC $\left(0 \leq x \leq \frac{l}{2}\right)$	$F_{NP} = 0$	$\bar{F}_N = 0$
	$F_{SP} = 0$	$\bar{F}_S = -1$
	$M_P = -\frac{q}{2} \left(x - \frac{l}{2}\right)^2$	$\bar{M} = -x$
Segment CB $\left(\frac{l}{2} \leq x \leq l\right)$	$F_{NP} = 0$	$\bar{F}_N = 0$
	$F_{SP} = -q \left(x - \frac{l}{2}\right)$	$\bar{F}_S = -1$

In segment AC, the internal forces under the actual load are all zero, so the integral is also zero.

In segment CB, the displacement caused by the bending deformation is:

$$\Delta_M = \int_{\frac{l}{2}}^l \frac{\bar{M} M_P}{EI} dx = \int_{\frac{l}{2}}^l -x \left[-\frac{q}{2} \left(x - \frac{l}{2}\right)^2 \right] \frac{dx}{EI} = \frac{q}{2EI} \cdot \frac{7l^4}{192} = \frac{7ql^4}{384EI} \quad (\downarrow)$$

The displacement caused by shear deformation is (for rectangular section, $k = 1.2$)

$$\Delta_S = \int_{\frac{l}{2}}^l \frac{k \bar{F}_S F_{SP}}{GA} dx = \int_{\frac{l}{2}}^l 1.2 \times (-1) \left[-q \left(x - \frac{l}{2} \right) \right] \frac{dx}{GA} = \frac{3ql^2}{20GA} \quad (\downarrow)$$

Since the axial force of the beam is zero, the total displacement is

$$\Delta = \Delta_M + \Delta_S = \frac{7ql^4}{384EI} + \frac{3ql^2}{20GA}$$

Now we compare the effects of shear deformation and bending deformation on the displacement. The ratio between them is:

$$\frac{\Delta_S}{\Delta_M} = \frac{\frac{3ql^2}{20GA}}{\frac{7ql^4}{384EI}} = 8.23 \frac{EI}{GA l^2}$$

Taking the Poisson's ratio of the material as $\mu = 1/3$, according to the mechanics of materials, $E/G = 2(1 + \mu) = 8/3$. For the rectangular section, $I/A = h^2/12$ (h is the height of the section). Substituting into the above formula, we have

$$\frac{\Delta_S}{\Delta_M} = 1.83 \left(\frac{h}{l} \right)^2$$

If the height-to-span ratio of the beam h/l is $1/10$, then $\Delta_S/\Delta_M = 1.83\%$. The effect of shear deformation is about 2% relative to the bending deformation, so generally the effect of shear deformation on the displacement can be ignored for beams. However, when the height-to-span ratio h/l increases to $1/3$, then Δ_S/Δ_M increases to $1/5$. Therefore, for deep beams, the effect of shear deformation on displacements cannot be neglected.

Example 4.2. As shown in figure 4.10a is a roof truss. The upper chords and other compression bars are made of reinforced concrete, and the lower chords and other tension bars are made of steel. Figure 4.10b is a simplified computational model of the roof truss subjected to uniformly distributed load q . Determine the vertical displacement of vertex C.

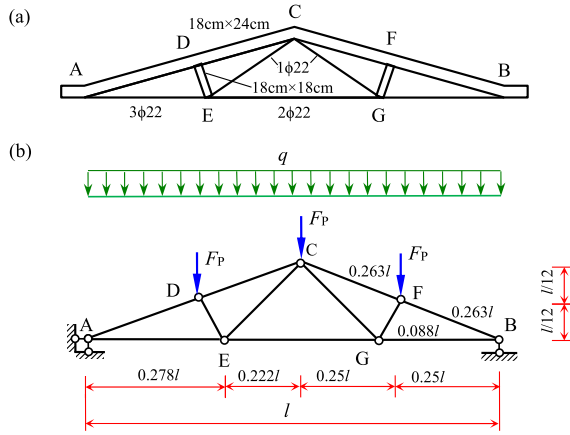
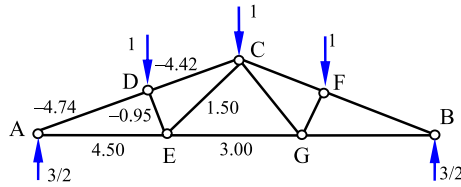
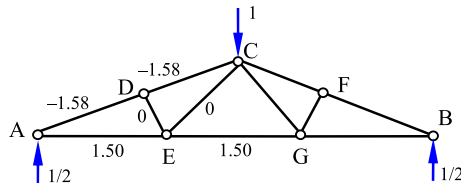
Solution: (1) Determine F_{NP} .

Firstly, the uniform load q can be converted into the nodal load $F_P = ql/4$ in terms of the average allocation rule along the horizontal length.

For the sake of simplification, the nodal load is taken as a unit value (figure 4.11). The axial forces F_{NP} can be obtained by multiplying the internal forces given in the figure by F_P .

(2) Determine $\bar{F}_N$.

Apply a virtual unit vertical load at point C, and the corresponding axial forces are shown in figure 4.12.

FIG. 4.10 – A roof truss under uniformly distributed load q .FIG. 4.11 – Determination of F_{NP} .FIG. 4.12 – Determination of $\bar{F}_N$.

(3) Determine Δ_C .

According to the displacement formula of trusses (4.21), we have

$$\Delta_C = \sum \frac{\bar{F}_N F_{NP} l}{EA}$$

The detailed calculation process is shown in table 4.2. Due to the symmetry, when calculating the sum, only half members of the truss are calculated in the table,

with member EG taking half-length. In table 4.2, A_h is the cross-sectional area of the upper chord: $A_h = 18 \text{ cm} \times 24 \text{ cm} = 432 \text{ cm}^2$. A_g is the cross-sectional area of the $\phi 22$ rebar: $A_g = 3.8 \text{ cm}^2$.

TAB. 4.2 – Tabulated calculation process for determining the displacement Δ_C .

Materials	Members	F_{NP}	l	A	$\bar{F}_N$	$\frac{\bar{F}_N F_{NP} l}{EA}$
Reinforced concrete	AD	$-4.74F_P$	$0.263l$	A_h	-1.58	$1.97F_P l / E_h A_h$
	DC	$-4.42F_P$	$0.263l$	A_h	-1.58	$1.84F_P l / E_h A_h$
	DE	$-0.95F_P$	$0.088l$	$0.75A_h$	0	0
						$\Sigma = \frac{3.81F_P l}{E_h A_h}$
Steel	CE	$1.50F_P$	$0.278l$	A_g	0	0
	AE	$4.50F_P$	$0.278l$	$3A_g$	1.50	$0.63F_P l / E_g A_g$
	EG	$3.00F_P$	$0.222l$	$2A_g$	1.50	$0.5F_P l / E_g A_g$
						$\Sigma = \frac{1.13F_P l}{E_g A_g}$

According to the table, we have

$$\Delta_C = 2F_P l \left(\frac{3.81}{E_h A_h} + \frac{1.13}{E_g A_g} \right) \quad (\text{a})$$

The original data are taken as follows:

Span $l = 12 \text{ m}$

Loads $q = 13\,000 \text{ N/m}$, $F_P = ql/4 = 39\,000 \text{ N}$

Elastic modulus of the concrete $E_h = 3.0 \times 10^4 \text{ MPa}$

Elastic modulus of the rebar $E_g = 2.0 \times 10^5 \text{ MPa}$

Substituting into equation (a), we get the displacement of vertex C

$$\Delta_C = 1.66 \text{ cm} (\downarrow)$$

Example 4.3. Figure 4.13a indicates a $1/4$ arc-shaped equal-section curved beam AB, with a radius of R and a rectangular cross-section. A concentrated load P is applied at point B vertically downward. Determine the vertical displacement Δ of point B.

Solution: To determine Δ , apply a virtual unit vertical load at point B (figure 4.13b), and solve the internal forces under the actual load and the virtual unit load, respectively. Take point B as the coordinate origin, and the central angle of an arbitrary point C is denoted as θ .

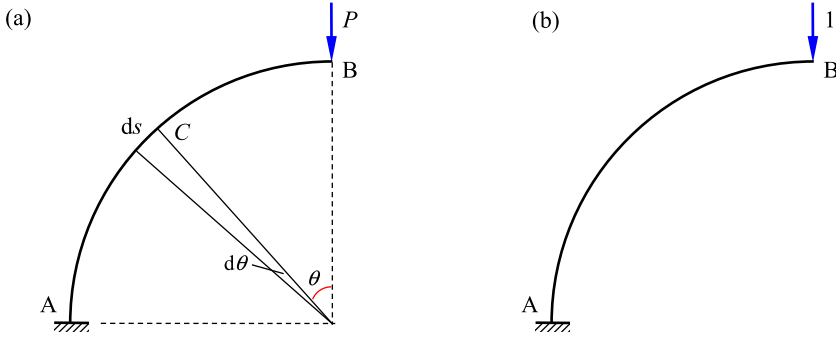


FIG. 4.13 – 1/4 arc-shaped curved beam under concentrated load.

Actual load	Virtual unit load
$M_P = -PR \sin \theta$	$\bar{M} = -R \sin \theta$
$F_{NP} = -P \sin \theta$	$\bar{F}_N = -\sin \theta$
$F_{SP} = P \cos \theta$	$\bar{F}_S = \cos \theta$

The displacement formula is

$$\Delta = \sum \int \frac{\bar{M} M_P}{EI} ds + \sum \int \frac{\bar{F}_N F_{NP}}{EA} ds + \sum \int \frac{k \bar{F}_S F_{SP}}{GA} ds$$

Using Δ_M , Δ_N , Δ_S to represent the displacements caused by M , F_N , F_S , respectively, we have

$$\begin{aligned} \Delta &= \left(\frac{PR^3}{EI} + \frac{PR}{EA} \right) \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta + \frac{kPR}{GA} \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= \frac{\pi PR^3}{4 EI} + \frac{\pi PR}{4 EA} + \frac{\pi kPR}{4 GA} \\ &= \Delta_M + \Delta_N + \Delta_S \end{aligned}$$

In example 4.1, the influence of bending deformation on displacement is dominant. In this example, the two ratios Δ_N/Δ_M and Δ_S/Δ_M are calculated for comparison.

Assume the curved beam is made of reinforced concrete, then $G \approx 0.4E$; the cross-section is a rectangle, $I/A = h^2/12$ (h is the height of the section), $k = 1.2$, thus

$$\frac{\Delta_N}{\Delta_M} = \frac{1}{12} \left(\frac{h}{R} \right)^2$$

$$\frac{\Delta_S}{\Delta_M} = k \frac{EI}{GAR^2} = \frac{1}{4} \left(\frac{h}{R} \right)^2$$

If h/R equals to 1/10, then $\Delta_N/\Delta_M = 1/1200$, $\Delta_S/\Delta_M = 1/400$. It can be seen that under the given conditions, the displacements caused by axial deformation and shear deformation are negligible compared to that caused by bending deformation.

Example 4.4. Figure 4.14a displays a simply supported beam subjected to a concentrated load F_P . Determine the relative rotation angle between end sections A and B of the beam.

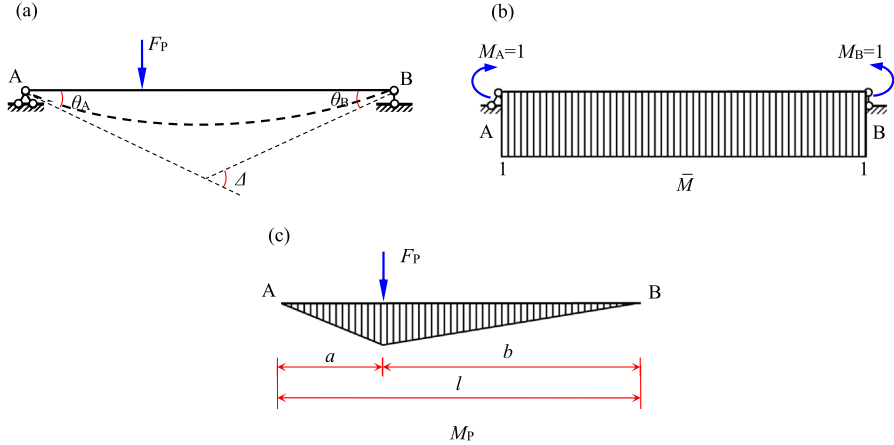


FIG. 4.14 – A simply supported beam under concentrated load.

Solution: As can be seen from figure 4.14a, the relative rotation angle $\Delta = \theta_A + \theta_B$. Therefore, the virtual force system is taken as a pair of reverse unit couples applied at sections A and B, as shown in figure 4.14b.

The moment of the beam under the actual load can be expressed as follows:

$$M_P = \frac{F_P b}{l} x \quad (0 < x < a)$$

$$M_P = F_P a \left(1 - \frac{x}{l}\right) \quad (a < x < l)$$

The moment diagram is shown in figure 4.14c.

Under the virtual unit load, the moment the beam is

$$\bar{M} = 1 \quad (0 < x < l)$$

Substituting into equation (4.20), the relative rotation angle Δ can be obtained:

$$\Delta = \sum \int \frac{\bar{M} M_P}{EI} ds = \int_0^a \frac{F_P b}{EI} x dx + \int_a^l \frac{F_P a}{EI} \left(1 - \frac{x}{l}\right) dx = \frac{F_P ab}{2EI} \quad (\curvearrowright)$$

The result is positive, indicating that the direction of the relative rotation angle Δ is the same as the applied unit couples.

4.5 Graph Multiplication Method

In section 4.4, the following integral needs to be solved when calculating the deflection of beams and frames under loads:

$$\int \frac{\overline{M} M_P}{EI} ds \quad (a)$$

When the number of structural members is large and the load situation is complicated, it is quite cumbersome to perform the analytical integral operation. Actual engineering structures are mostly composed of uniform straight members. In this case, the fast and convenient graph multiplication method can be employed to obtain an accurate integral solution, thus simplifying the calculation.

4.5.1 Graph Multiplication Method and Its Application Conditions

As shown in figure 4.15, for the two moment diagrams of a uniform straight bar AB, assume one of the moment diagrams ($\overline{M}$) is a straight line, and the other one (M_P) is an arbitrary shape. Taking the bar axis as the x -axis, the intersection O of the extension line of diagram $\overline{M}$ and the x -axis as the coordinate origin. Assume the slope of the inclined line in the diagram $\overline{M}$ is α , then the vertical distance (ordinate) of any point on the diagram $\overline{M}$ can be written as

$$\overline{M} = x \tan \alpha \quad (b)$$

Substituting equation (b) into the displacement formula (a) of beams and frames, we have

$$\Delta = \int \frac{\overline{M} M_P}{EI} ds = \tan \alpha \int_A^B \frac{x M_P}{EI} dx \quad (c)$$

where, $M_P dx$ can be regarded as the differential area of the diagram $\overline{M}$ (shaded area in figure 4.15); $x \cdot M_P dx$ is the area moment of the differential area with respect to the y -axis. Therefore, $\int_A^B x M_P dx$ is the area moment of area A of the diagram M_P with respect to the y -axis. If the distance from the centroid C of the diagram M_P to the y -axis is x_C , then

$$\int_A^B x M_P dx = A x_C \quad (d)$$

If the bending stiffness EI of the member section is constant in segment AB, substituting equation (d) into equation (c), we have

$$\Delta = \frac{1}{EI} \tan \alpha \int_A^B x M_P dx = \frac{1}{EI} \tan \alpha \cdot A x_C = \frac{1}{EI} A y_C \quad (4.24)$$

where, y_C is the ordinate in the diagram $\overline{M}$ corresponding to the centroid C of the diagram M_P , and A is the area of the diagram M_P in segment AB.

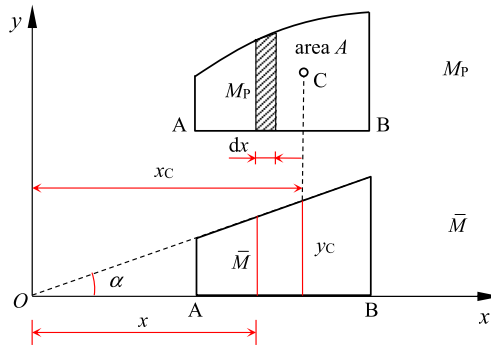


FIG. 4.15 – The principle of graph multiplication method.

Therefore, the integral operation of structural displacement can be transformed into the product operation between the area and ordinate of the two moment diagrams. This is called the **graph multiplication method**. Equation (4.24) is the formula of the graph multiplication method for calculating the displacements of a uniform straight bar.

If all the bending members in a structure can be multiplied with graphs, the displacement formula can be written as

$$\Delta = \sum \frac{1}{EI} A y_C \quad (4.25)$$

According to the above derivation process, the following two points should be noted when using the graph multiplication method:

- (1) Application conditions: The member should be a uniform straight member; at least one of the two moment diagrams should be a straight line; the ordinate y_C can only be taken from the straight-line diagram.
- (2) Sign convention: When area A and ordinate y_C are on the same side of the member, the product $A y_C$ should be positive; otherwise negative.

The area formulas and centroid positions of several common moment diagrams for deflection calculation are given in figure 4.16.

It should be pointed out that in each of the parabolic diagrams, the tangent at the vertex of the parabola is parallel to the baseline (the member axis). These diagrams are referred to as standard parabolic diagrams. When using the formulas in the figure, we should note whether it is a standard diagram.

4.5.2 Several Specific Problems of Applying Graph Multiplication Method

- (1) If both diagrams are straight lines, the ordinate y_C can be taken from any of the diagrams.

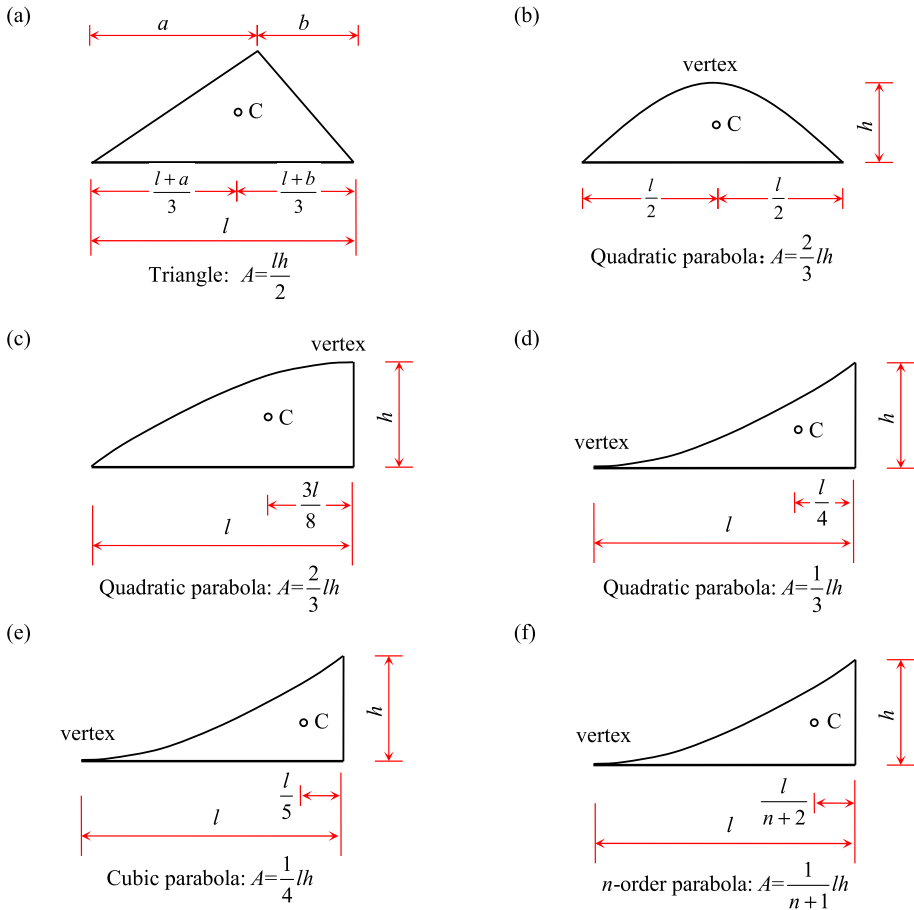


FIG. 4.16 – The area formulas and centroid positions of several common moment diagrams.

- (2) If one diagram is a polyline consisting of several straight lines, and the other diagram is a curve, it can be considered piecewise. Taking the situation shown in figure 4.17 as an example, we have

$$\frac{1}{EI} \int \overline{M} M_P ds = \frac{1}{EI} (A_1 y_1 + A_2 y_2 + A_3 y_3)$$

- (3) When the diagram is complicated and its area or centroid position is not easy to determine, it can be decomposed into several simple diagrams according to the rules of integral operation. They can be multiplied by the other diagram respectively, and then the algebraic sum is the result of the multiplication of the two diagrams.

For instance, consider the multiplication of two trapezoids displayed in figure 4.18. Instead of determining the area and centroid of the trapezoids, one of

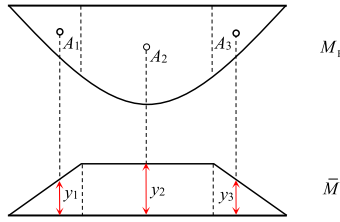


FIG. 4.17 – Illustration of piecewise graph multiplication.

the trapezoids (the diagram M_P) can be decomposed into two triangles (or a rectangle and a triangle). Then by applying the graph multiplication method, we have

$$\frac{1}{EI} \int \bar{M} M_P ds = \frac{1}{EI} (A_1 y_1 + A_2 y_2) \quad (e)$$

in which,

$$A_1 = \frac{1}{2} al, \quad A_2 = \frac{1}{2} bl$$

$$y_1 = \frac{2}{3} c + \frac{1}{3} d, \quad y_2 = \frac{1}{3} c + \frac{2}{3} d$$

Substituting into equation (e), we have

$$\frac{1}{EI} \int \bar{M} M_P ds = \frac{l}{6EI} (2ac + 2bd + ad + bc) \quad (4.26)$$

Equation (4.26) is the deflection calculation formula for the graph multiplication of two trapezoids. The sign of each item in the parentheses should follow the rule that the multiplication of the ordinate on the same side of the baseline is positive and otherwise negative. Equation (4.26) is also applicable to the case where the ordinate on one end is zero, *i.e.*, the case of triangles.

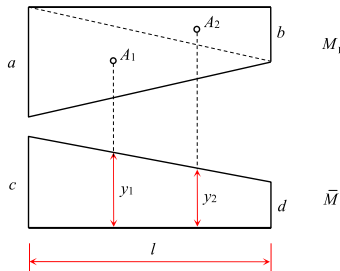


FIG. 4.18 – Graph multiplication of two trapezoids.

In addition, if the parabola is a non-standard diagram, it can be transformed into a standard one.

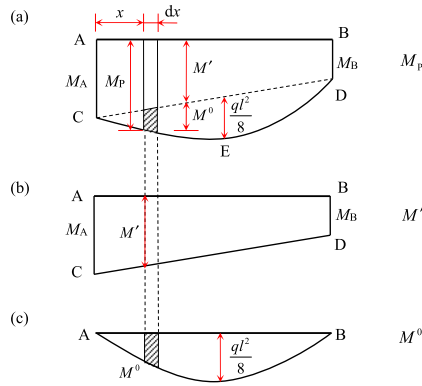


FIG. 4.19 – Graph multiplication of non-standard parabola diagram.

Figure 4.19a shows the diagram M_P of a straight member AB in a structure subjected to a uniformly distributed load q , which is a non-standard parabolic diagram. When calculating by the graph multiplication method, the diagram M_P can be decomposed into two diagrams: a trapezoidal diagram whose moments at two ends are M_A and M_B (the diagram M' in figure 4.19b), and a standard quadratic parabolic diagram whose moment is that of the corresponding simply supported beam under the uniformly distributed load q (the diagram M^0 in figure 4.19c). Therefore, by multiplying the above two diagrams with the diagram $\bar{M}$ respectively, their algebraic sum is the desired result.

It is worth noting that the so-called superposition of moment diagrams refers to the superposition of their ordinates. Although the superimposed parabolic diagram is not the same in shape as the original standard parabola, their ordinate is equal at the same abscissa x , and the area of the infinitesimal element dx (the shaded area in the figure) is also the same. Therefore, the area and centroid position of both diagrams is the same. Understanding this point is conducive to the decomposition of complex moment diagrams.

4.5.3 Examples of Graph Multiplication Method

Example 4.5. Figure 4.20a illustrates a cantilever beam subjected to a concentrated load F_P at point A. Determine the deflection Δ_C at the midpoint C.

Solution: To determine the deflection at the midpoint C, apply a downward virtual unit load at point C. Draw the diagrams M_P and $\bar{M}$, as shown in figure 4.20a, b. The area of the triangle in the diagram $\bar{M}$ is

$$A = \frac{1}{2} \times \frac{l}{2} \times \frac{l}{2} = \frac{l^2}{8}$$

The corresponding ordinate in the diagram M_P is

$$y_0 = \frac{5}{6} F_P l$$

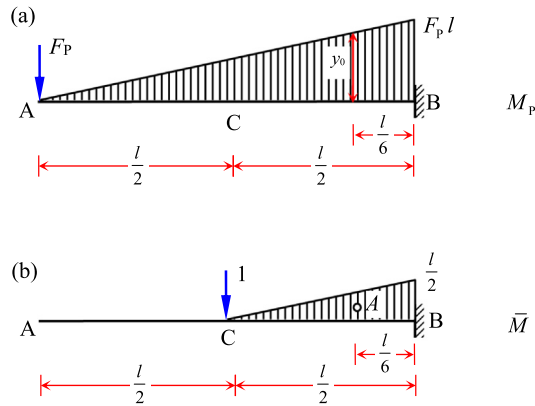


FIG. 4.20 – Deflection calculation of a cantilever beam by graph multiplication method.

According to the graph multiplication method, we have

$$\Delta_C = \frac{1}{EI} \int \bar{M} M_P ds = \frac{1}{EI} \times \frac{l^2}{8} \times \frac{5}{6} F_P l = \frac{5 F_P l^3}{48 EI} \quad (\downarrow)$$

Example 4.6. Determine the horizontal displacement Δ at point B of the frame shown in figure 4.21a. For each member, the cross-section is a rectangle of $b \times h$, and the bending stiffness is EI . Only the influence of bending deformation needs to be considered in the calculation.

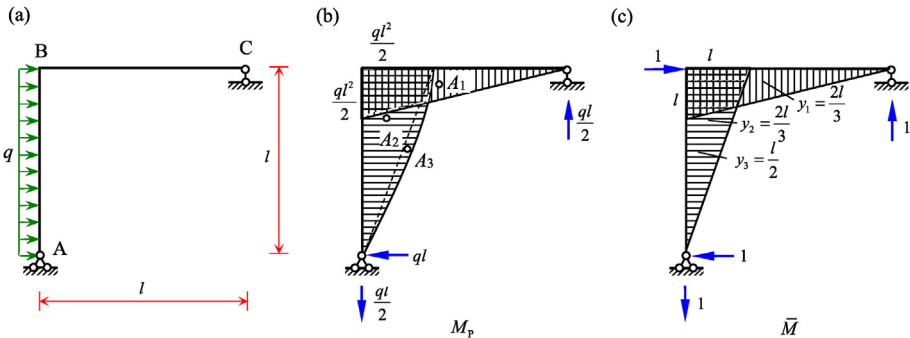


FIG. 4.21 – Deflection calculation of a frame by graph multiplication method.

Solution: To determine the horizontal displacement of point B, apply a rightward virtual unit load at point B. Draw the diagrams M_P and $\bar{M}$, as illustrated in figure 4.21b, c.

The area of the diagram M_P can be divided into three parts, A_1 , A_2 , A_3 :

$$A_1 = \frac{1}{2} \times \frac{ql^2}{2} \times l = \frac{ql^3}{4}, \quad A_2 = \frac{ql^3}{4}, \quad A_3 = \frac{2}{3} \times \frac{ql^2}{8} \times l = \frac{ql^3}{12}$$

The corresponding ordinates in the diagram $\bar{M}$ are:

$$y_1 = y_2 = \frac{2l}{3}, \quad y_3 = \frac{l}{2}$$

Thus

$$\Delta = \frac{1}{EI} \int \bar{M} M_P ds = \frac{1}{EI} \times \left(\frac{ql^3}{4} \times \frac{2l}{3} + \frac{ql^3}{4} \times \frac{2l}{3} + \frac{ql^3}{12} \times \frac{l}{2} \right) = \frac{3ql^4}{8EI} (\rightarrow)$$

Example 4.7. Figure 4.22a shows a cantilever beam of variable stiffness subjected to a uniformly distributed load q . Determine the vertical displacement Δ_{yB} of point B.

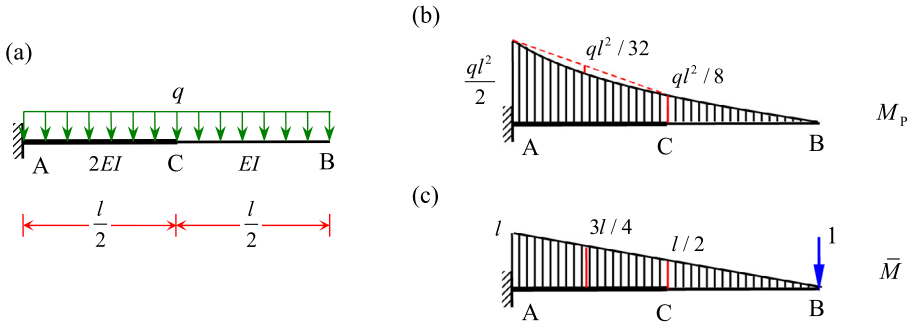


FIG. 4.22 – Deflection calculation of a cantilever beam of variable stiffness by graph multiplication method.

Solution: To determine the vertical displacement of point B, apply a downward virtual unit load at point B. Draw the diagrams M_P and $\bar{M}$, as shown in figure 4.22b, c. Due to the different stiffness of segments AC and CB, when applying the graph multiplication method, it should be treated segmentally. The left half of the diagram M_P is a non-standard parabola, which can be regarded as a trapezoid minus a standard parabola, as shown in figure 4.22b.

Therefore

$$\begin{aligned} \Delta_{yB} &= \frac{1}{EI} \times \left(\frac{1}{3} \times \frac{ql^2}{8} \times \frac{l}{2} \right) \times \left(\frac{3}{4} \times \frac{l}{2} \right) \\ &\quad + \frac{1}{2EI} \times \frac{l/2}{6} \times \left(2 \times \frac{ql^2}{2} \times l + 2 \times \frac{ql^2}{8} \times \frac{l}{2} + \frac{ql^2}{2} \times \frac{l}{2} + \frac{ql^2}{8} \times l \right) \\ &\quad - \frac{1}{2EI} \times \left(\frac{2}{3} \times \frac{ql^2}{32} \times \frac{l}{2} \right) \times \frac{3l}{4} = \frac{17ql^4}{256EI} (\downarrow) \end{aligned}$$

Here, the formula (4.26) for the multiplication of two trapezoids is used during the calculation process. For this problem, there are several ways of performing graph multiplication, and it is also convenient to utilize the analytical integral method to compute the vertical displacement at point B.

Example 4.8. Determine the relative horizontal displacement Δ_{AB} between points A and B on the frame in figure 4.23a.

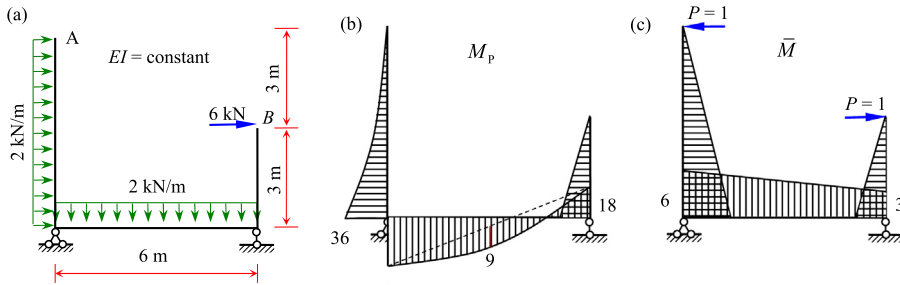


FIG. 4.23 – Relative displacement calculation of a frame by graph multiplication method.

Solution: To determine the relative horizontal displacement between points A and B, apply a pair of horizontal unit loads in opposite direction at points A and B. Draw the diagrams M_P and $\bar{M}$, as shown in figure 4.23b, c.

According to the graph multiplication method, we have

$$\begin{aligned} \Delta_{AB} = & \frac{1}{EI} \left[\frac{6}{6} \times (-2 \times 36 \times 6 + 2 \times 18 \times 3 - 36 \times 3 + 18 \times 6) - \frac{2}{3} \times 6 \times 9 \times \frac{3+6}{2} \right] \\ & + \frac{1}{EI} \left(-\frac{1}{3} \times 36 \times 6 \times \frac{3}{4} \times 6 + \frac{18 \times 3}{2} \times \frac{2}{3} \times 3 \right) = \frac{-756}{EI} (\rightarrow \leftarrow) \end{aligned}$$

4.6 Deflection Calculation under Temperature Change

When the temperature of a statically determinate structure changes, deformation, and displacements will occur in the structure due to thermal expansion and contraction of the material. However, the members of a statically determinate structure can deform freely, so no internal force will generate in the structure. In this section, the deflection calculation of statically determinate structures under temperature changes will be discussed.

As shown in figure 4.24a, assume the outside temperature of the structural members is increased by t_1 and the inside temperature is increased by t_2 . To simplify the calculation, it is assumed that the temperature varies linearly along the height of the member section. Under this condition, the deformation of the differential element of the members is displayed *via* the dashed line in figure 4.24b, where the section remains a plane after deformation. It can be seen that the deformation of the

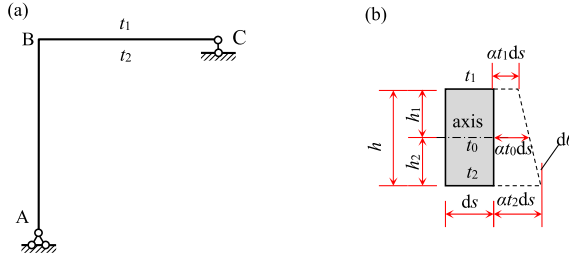


FIG. 4.24 – (a) A frame subjected to temperature change, (b) deformation of differential element.

members caused by temperature change can be decomposed into two parts: the expansion or contraction deformation along the member axis and the relative rotation of the section. No shear deformation is caused in the members.

In terms of the geometric relation, temperature change at the neutral axis of members is

$$t_0 = \frac{h_1 t_2 + h_2 t_1}{h}$$

where, h denotes the member section height, h_1 and h_2 are the distances from the member neutral axis to the upper and lower edges, respectively.

If the member section is symmetrical to the neutral axis, *i.e.*, $h_1 = h_2 = \frac{1}{2}h$, then the above formula becomes

$$t_0 = \frac{t_1 + t_2}{2}$$

If the linear expansion coefficient of the material is α , the axial strain and curvature of the infinitesimal element caused by temperature change can be expressed as

$$\varepsilon = \alpha t_0 \quad (a)$$

$$\kappa = \frac{d\theta}{ds} = \frac{\alpha(t_2 - t_1)ds}{hds} = \frac{\alpha\Delta t}{h} \quad (b)$$

where, the temperature difference between the upper and lower edges of members is $\Delta t = t_2 - t_1$. Substituting equations (a) and (b) into equation (4.16), and noticing that the average shear strain $\gamma_0 = 0$, we have

$$\Delta = \sum \int \bar{F}_N \alpha t_0 ds + \sum \int \bar{M} \frac{\alpha \Delta t}{h} ds \quad (4.27)$$

The above formulation is a general formula for calculating the displacement of statically determinate structures under temperature change. The integral operation is performed along the entire length of the members, and the sum operation is

performed for each member of the structure. The first term on the right side of the equation represents the displacement caused by the average temperature change, and the second term represents the displacement caused by the difference in temperature change between the upper and lower edges of the members.

If the section height of members does not change and the temperature change along the member length is the same, the above formula can be rewritten as

$$\Delta = \sum \alpha t_0 \int \bar{F}_N ds + \sum \frac{\alpha \Delta t}{h} \int \bar{M} ds \quad (4.28)$$

When applying equation (4.27) or (4.28), we should pay attention to the sign of the terms on the right side. The axial force $\bar{F}_N$ is positive for tension, and t_0 is positive when the temperature increases. When the bending directions caused by the moment $\bar{M}$ and temperature difference Δt are the same (that is, $\bar{M}$ and Δt cause the members to generate tensile deformation in the same side's fibers), their product has positive value. Otherwise, it is negative.

For beams and frames, generally, the influence of axial deformation cannot be ignored when computing the displacement caused by temperature change.

For trusses, the deflection calculation formula under temperature change is expressed as

$$\Delta = \sum \bar{F}_N \alpha t_0 l \quad (4.29)$$

Example 4.9. Figure 4.25a shows a frame, the temperature of the lower side of the beam and the right side of the column increases by 10°C , and the temperature of the upper side of the beam and the left side of the column keeps unchanged. Determine the vertical displacement at point C. The cross-section of each member is a rectangle, with a section height of $h = 60\text{ cm}$, $a = 6\text{ m}$, $\alpha = 10^{-5}^\circ\text{C}^{-1}$.

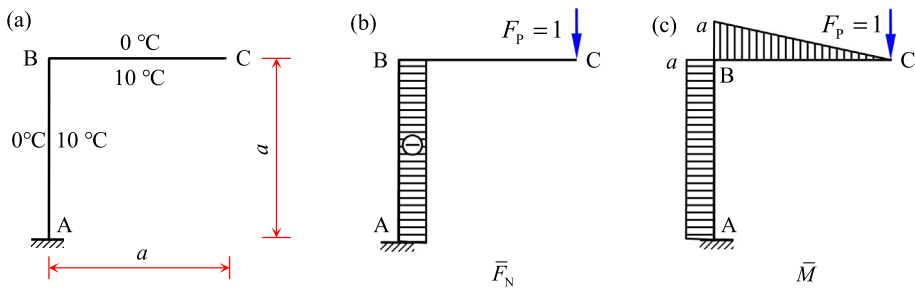


FIG. 4.25 – Deflection calculation of a frame under temperature change.

Solution: To determine the vertical displacement of point C, apply an upward virtual unit load at point C. Draw the diagrams $\bar{F}_N$ and $\bar{M}$, as shown in figure 4.25b, c.

The temperature rise at the member axis is

$$t_0 = \frac{10^\circ\text{C} + 0^\circ\text{C}}{2} = 5^\circ\text{C}$$

The temperature difference between the upper and lower (left and right) edges is

$$\Delta t = 10^\circ\text{C} - 0^\circ\text{C} = 10^\circ\text{C}$$

Substituting into equation (4.28), we have

$$\begin{aligned}\Delta_C &= \sum \alpha t_0 \int \bar{F}_N ds + \sum \frac{\alpha \Delta t}{h} \int \bar{M} ds \\ &= 5\alpha \times (-a) - \frac{10\alpha}{h} \times \frac{3}{2} a^2 = -5\alpha a \times \left(1 + \frac{3a}{h}\right)\end{aligned}$$

Since the bending directions caused by Δt and $\bar{M}$ are inverse, the second term in the above formula takes a negative value. Substituting $\alpha = 10^{-5}^\circ\text{C}^{-1}$, $a = 600\text{ cm}$, $h = 60\text{ cm}$, we have

$$\Delta_C = -0.93\text{ cm } (\uparrow)$$

4.7 Reciprocal Theorems of Linearly Elastic Structures

In this section, four universal theorems of linearly elastic structures, *i.e.*, the **reciprocal theorems**, are discussed. The most basic theorem is the theorem of reciprocal works, and the other three theorems can be derived from it. These theorems will be used in the chapters on the force method and displacement method in this textbook.

The reciprocal theorems are only applicable to linearly deformable structures. The application conditions are:

- (1) The material is in the elastic phase and the stress is proportional to strain.
- (2) The structural deformation is small and does not affect the external forces applying to the structures.

4.7.1 Theorem of Reciprocal Works

Figure 4.26a, b demonstrate two states when a linearly elastic structure is subjected to external forces F_{Pi} and F_{Pj} , which are referred to as state i and state j respectively. Now we consider the work done by the two forces when they are applied to the structure in different orders. If F_{Pi} is applied first and then F_{Pj} is applied, the deformation state of the structure is shown in figure 4.26c. The total work done by the external forces is

$$W_1 = \frac{1}{2} F_{Pi} \Delta_{ii} + F_{Pi} \Delta_{ij} + \frac{1}{2} F_{Pj} \Delta_{jj}$$

where, for displacement Δ_{ij} , the first subscript i indicates the location and direction of the displacement, that is, the displacement of point i in the F_{Pi} direction; the second subscript j denotes the cause of displacement, that is, the displacement is caused by F_{Pj} .

If F_{Pj} is applied first and then F_{Pi} is applied, the deformation state of the structure is illustrated in figure 4.26d. The total work done by the external forces is

$$W_2 = \frac{1}{2} F_{Pj} \Delta_{jj} + F_{Pj} \Delta_{ji} + \frac{1}{2} F_{Pi} \Delta_{ii}$$

In the above two loading processes, although the loading order of external forces is different, the final load state is the same. According to the uniqueness theorem of the solution of linear elastic structures, the final deformation state of the structure should be identical. Therefore, there should be identical strain energy in the structure for the two loading processes. Owing to the conservation of energy, the total work done by the external forces in the above two loading processes should be identical, that is, the total work done by the external forces is independent of the loading order. Therefore, we have:

$$W_1 = W_2$$

That is

$$\frac{1}{2} F_{Pi} \Delta_{ii} + F_{Pi} \Delta_{ij} + \frac{1}{2} F_{Pj} \Delta_{jj} = \frac{1}{2} F_{Pj} \Delta_{jj} + F_{Pj} \Delta_{ji} + \frac{1}{2} F_{Pi} \Delta_{ii}$$

Therefore

$$F_{Pi} \Delta_{ij} = F_{Pj} \Delta_{ji} = W \quad (4.30)$$

The above equation shows that the work done by the external forces of state i on the displacements of state j is equal to the work done by the external forces of state j on the displacements of state i . This is the **theorem of reciprocal works**. Because the two states of the equilibrium force system and compatible deformation are independent, the theorem of reciprocal works is actually the theorem of reciprocal virtual works.

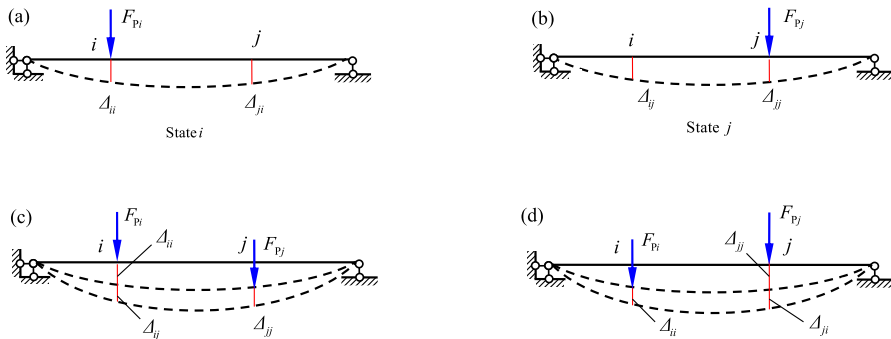


FIG. 4.26 – The schematic of the theorem of reciprocal works.

It should be noted that the superposition principle is adopted in the above derivation process, that is, the displacement of the structure under the combined action of F_{Pi} and F_{Pj} is considered to be identical to the sum of the displacements caused by them separately. This makes the theorem of reciprocal works only applicable to linearly elastic structures. In addition, F_{Pi} and F_{Pj} can be generalized forces, including a set of external forces, then the displacements Δ_{ij} and Δ_{ji} are the corresponding generalized displacements. Besides the member structures, the theorem of reciprocal works is also applicable to linearly elastic continuum structures, such as plates, shells, and massive structures.

4.7.2 Theorem of Reciprocal Displacements

Now we apply the theorem of reciprocal works to study a special case. If the forces acting on the structure are all unit loads with the dimension as 1, *i.e.*, $F_{Pi} = F_{Pj} = 1$, as shown in figure 4.27. Using δ to represent the displacement caused by the unit load, according to the theorem of reciprocal works, we have

$$1 \cdot \delta_{ij} = 1 \cdot \delta_{ji}$$

That is

$$\delta_{ij} = \delta_{ji} \quad (4.31)$$

This is the **theorem of reciprocal displacements**. It indicates that the displacement corresponding to the i -th unit load caused by the j -th unit load is identical to the displacement corresponding to the j -th unit load caused by the i -th unit load. $\delta_{ij} = \frac{\Delta_{ij}}{F_{Pj}}$ is called the displacement influence coefficient, whose dimension is that of $\frac{W}{F_{Pi}F_{Pj}}$. Obviously, the theorem of reciprocal displacements is a special form of the theorem of reciprocal works when $F_{Pi} = F_{Pj} = 1$.



FIG. 4.27 – The schematic of the theorem of reciprocal displacements.

The loads here can also be generalized loads, and then the displacements are the corresponding generalized displacements. For instance, in the two states of figure 4.28, according to the theorem of reciprocal displacements, we have $\theta_A = \delta_C$. In fact, according to the mechanics of materials:

$$\theta_A = \frac{Fl^2}{16EI}, \quad \delta_C = \frac{Mt^2}{16EI}$$

Now $F = 1$, $M = 1$, and then $\theta_A = \delta_C = \frac{l^2}{16EI}$. Therefore, although θ_A represents the angular displacement caused by the unit load, δ_C indicates the linear displacement caused by the unit couple, their meanings are different, but they have identical values and dimensions (*i.e.*, the dimension of $W/(FM)$). In other words, the dimensions of two generalized displacements may be different, but their influence coefficients can be the same in value and dimension. Therefore, strictly speaking, the theorem of reciprocal displacements should be named the theorem of reciprocal displacement influence coefficients. But customarily, it is still called the theorem of reciprocal displacements.

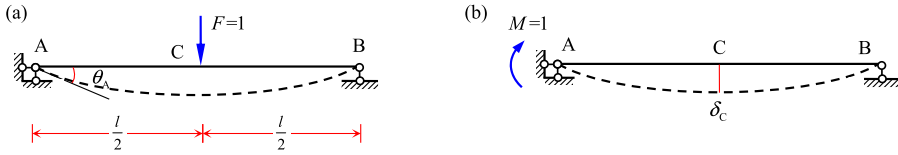


FIG. 4.28 – The schematic of reciprocal displacement influence coefficients.

4.7.3 Theorem of Reciprocal Reactions

The theorem of reciprocal reactions is also a special form of the theorem of reciprocal works. It shows that when two supports of a statically indeterminate structure have a unit displacement respectively, the reactions caused by them have a reciprocal relation.

Figure 4.29 demonstrates two deformation states of a linear elastic structure. In figure 4.29a, due to the unit displacement $\Delta_i = 1$ of support i , the reactions caused at supports i and j are r_{ii} and r_{ji} , respectively; in figure 4.29b, due to the unit displacement $\Delta_j = 1$ of support j , the reactions caused at supports i and j are r_{ij} and r_{jj} , respectively. Herein, among the two subscripts of reaction r_{ij} , the first subscript i indicates that the reaction corresponds to the unit displacement Δ_i , and the second subscript j indicates that the reaction is caused by the unit displacement Δ_j .

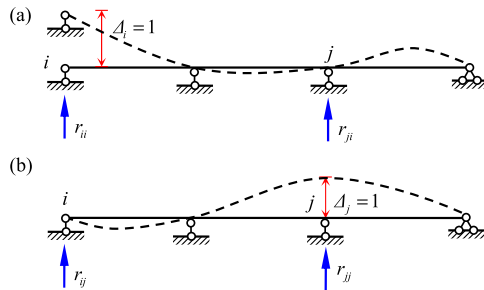


FIG. 4.29 – The schematic of the theorem of reciprocal reactions.

Applying the theorem of reciprocal works to the above two states, we have

$$r_{ii} \times 0 + r_{ji} \times 1 = r_{ij} \times 1 + r_{jj} \times 0$$

i.e.,

$$r_{ij} = r_{ji} \quad (4.32)$$

This is the **theorem of reciprocal reactions**, which means that the reaction corresponding to the i -th unit support displacement caused by the j -th unit support displacement is identical to the reaction corresponding to the j -th unit support displacement caused by the i -th unit support displacement. r_{ij} and r_{ji} are called the reaction influence coefficients, whose value and dimension (*i.e.*, the dimension of $\frac{W}{A_i A_j}$) are identical. This theorem is applicable to any two supports in the structure. Nevertheless, we should pay attention to the correspondence between the reaction and displacement in the work relation, that is, a force corresponds to a linear displacement, and a couple corresponds to an angular displacement.

4.7.4 Theorem of Reciprocal Displacement-Reaction

The theorem of reciprocal displacement-reaction is another special form of the theorem of reciprocal works. For example, figure 4.30 shows two deformed states of a linear elastic structure. When a unit load $F_{Pi} = 1$ is applied at section i , the reaction moment at support j is r'_{ji} , whose direction is shown in figure 4.30a. Then, when a unit rotation $\theta_j = 1$ occurs at support j in the direction of r'_{ji} , the displacement of section i in the direction of F_{Pi} is denoted as δ'_{ij} , as displayed in figure 4.30b.

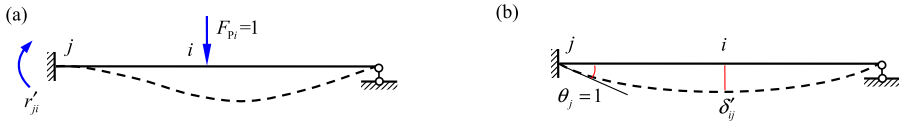


FIG. 4.30 – The schematic of the theorem of reciprocal displacement-reaction.

Applying the theorem of reciprocal works to the above two states, we have

$$r'_{ji} \times 1 + 1 \times \delta'_{ij} = 0$$

i.e.

$$\delta'_{ij} = -r'_{ji} \quad (4.33)$$

This is the **theorem of reciprocal displacement-reaction**, which means that the displacement corresponding to the i -th unit load caused by the j -th unit support displacement is equal to the reaction corresponding to the j -th unit support displacement caused by the i -th unit load, but the sign is opposite. The value and dimension of the two influence coefficients in equation (4.33) are the same. The force here can also be a generalized force, and then the displacement is the corresponding generalized displacement.

Problems

4.1 Determine the horizontal displacement of point B of the truss. Each member has an identical EA .

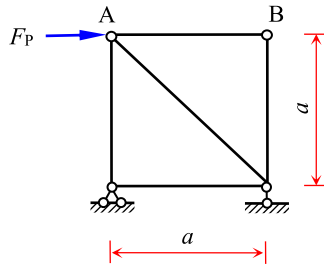


FIG. P4.1

4.2 Determine the vertical displacement of point C of the truss. Each member has an identical EA .

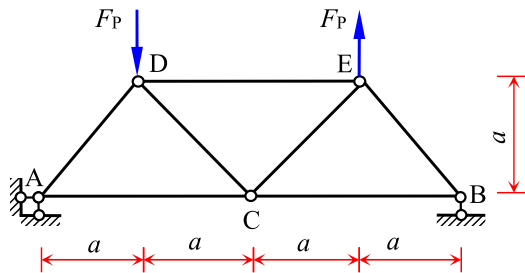


FIG. P4.2

4.3 Determine the relative linear displacement between points A and B of the truss. Each member has an identical EA .

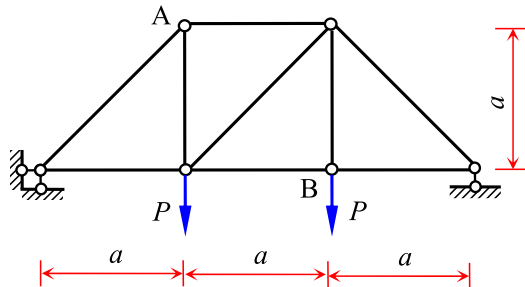


FIG. P4.3

4.4 As shown in the figure is a plane truss, each member has an identical cross-sectional area $A = 0.4 \times 10^{-2} \text{ m}^2$ and elastic modulus $E = 200 \text{ GPa}$. Determine the vertical displacement of points B and D.

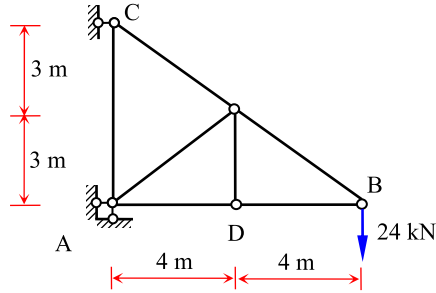


FIG. P4.4

4.5 Determine the mid-span deflection of the beam by the integral method (ignore the influence of shear deformation).

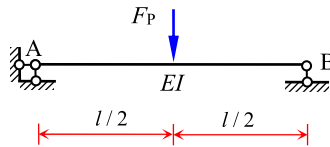


FIG. P4.5

4.6 Determine the vertical displacement Δ_C of midpoint C of the simply supported beam, and compare the influence of shear force and moment on the displacement. The cross-section is a rectangle with height h , $G = \frac{3}{8}E$, $k = 1.2$, $\frac{h}{l} = \frac{1}{10}$.

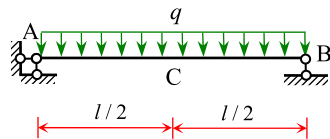


FIG. P4.6

4.7 Determine the vertical displacement of point B of the curved beam by the integral method. $EI = \text{constant}$.

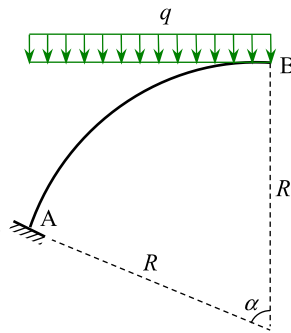


FIG. P4.7

4.8 Determine the horizontal displacement Δ_{xB} of point B of the curved beam. The axis of the curved beam is a parabola, whose equation is

$$y = \frac{4f}{l^2} x(l - x)$$

The beam is subjected to a uniformly distributed load q , $EI = \text{constant}$. In the calculation, only the bending deformation is needed to consider. The arch is relatively flat, and it can be regarded that $ds = dx$.

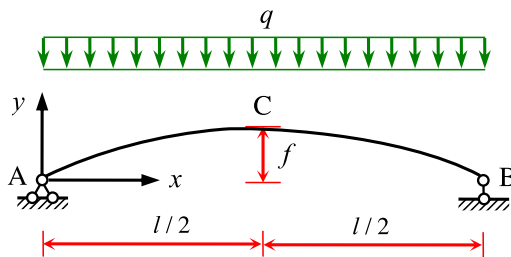


FIG. P4.8

4.9 Determine the maximum deflection $f_{\max}$ of the beam by the graph multiplication method.

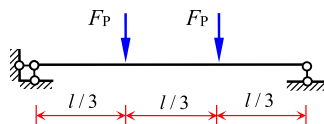


FIG. P4.9

4.10 Determine the deflection at point C of the beam, where, $F_P = 9000 \text{ N}$, $q = 15\,000 \text{ N/m}$. The beam is #18 steel I-beam, $I = 1660 \text{ cm}^4$, $h = 18 \text{ cm}$, $E = 2.1 \times 10^5 \text{ MPa}$.

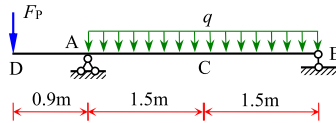


FIG. P4.10

4.11 Determine the horizontal displacement at point B in the frame structure shown in the figure.

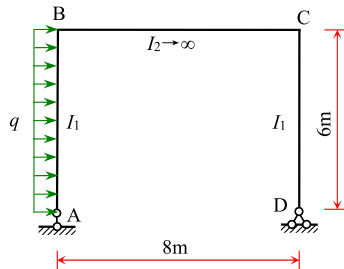


FIG. P4.11

4.12 Determine the vertical displacement at point C of the frame. $EI = \text{constant}$.

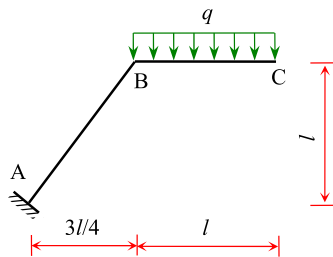


FIG. P4.12

4.13 Determine the vertical displacement at point D of the frame. $EI = \text{constant}$.

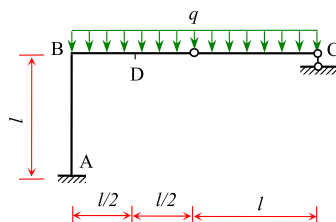


FIG. P4.13

4.14 Determine the vertical displacement of point C of the frame. $EI = \text{constant}$.

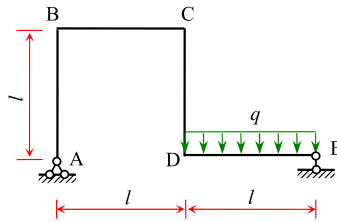


FIG. P4.14

4.15 The temperature of members AD and AC of the truss shown in the figure increases by t °C, and the coefficient of linear expansion of the material is α . Determine the vertical displacement of point C.

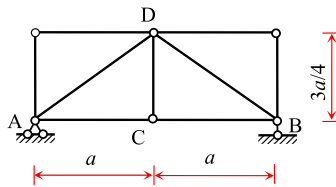


FIG. P4.15

4.16 The frame shown in the figure has a temperature increase of 30 °C on the inner side, each member has a rectangular cross-section with the same height of h . Determine the vertical displacement of point C.

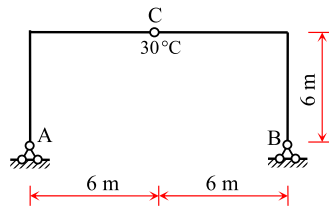


FIG. P4.16

4.17 The frame shown in the figure has a rectangular cross-section with height h , $h/l = 1/20$. Determine the vertical displacement of point C.

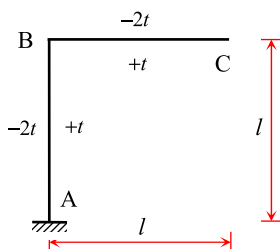


FIG. P4.17

4.18 A pair of couple moments M are applied at both ends of the simply supported beam shown in the figure. Meanwhile, the temperature of the upper surface rises by t_1 , and the lower surface drops by t_1 . Determine the rotation angle θ of the endpoint. If $\theta = 0$, how much is the couple moment M ? The beam has a rectangular cross-section with a dimension of $b \times h$.

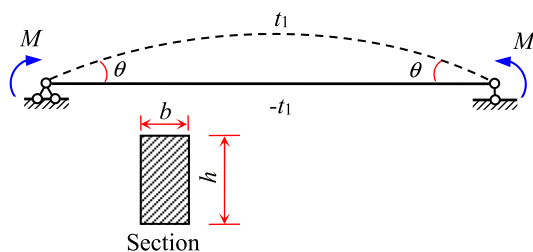


FIG. P4.18

4.19 It is known that the deflection curve equation of a uniform simply supported beam with a concentrated load F_P acting on the mid-span (as shown in figure P4.19a) is $y(x) = \frac{F_P x}{48EI} (3l^2 - 4x^2)$, $0 \leq x \leq \frac{l}{2}$. Determine the mid-span deflection Δ of the beam under the uniformly distributed load q shown in figure P4.19b by the theorem of reciprocal works.

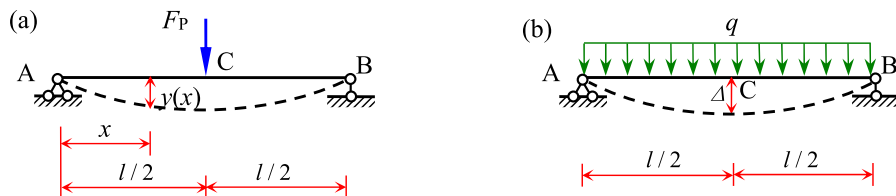


FIG. P4.19

Chapter 5

Force Method

Compared with statically determinate structures, statically indeterminate structures have redundant constraints, making them more resistant to damage. Therefore, they are more widely used in engineering practice. The main difference between statically indeterminate and determinate structures lies in that the internal forces of statically determinated structures can be obtained directly from the static equilibrium condition, without considering the **compatibility condition of deformation**. Accordingly, their internal forces are statically determinate. While for statically indeterminate structures, the internal forces cannot be obtained directly from the static equilibrium condition and it is necessary to consider the compatibility condition of deformation. Thus, the corresponding internal forces are statically indeterminate. The main content of structural analysis includes the calculation of internal forces and displacements, so as to check the strength and stiffness. The conditions for solving the internal forces and deflections of statically indeterminate structures contain the static equilibrium condition, compatibility condition of deformation, and physical condition.

According to a different selection of primary unknowns, the analysis methods of statically indeterminate structures can be divided into two types: the force method and the displacement method. The former utilizes the redundant constraint forces as the primary unknowns and the latter adopts the independent nodal displacements. The **force method** is an earlier proposed and well-developed method for the analysis of statically indeterminate structures, which is also a more fundamental method. When solving the internal forces of a statically indeterminate structure by the force method, the primary unknowns are the redundant constraint forces. The primary system and primary structure should be first selected to establish the force method equations from the compatibility condition of deformation. Then, the displacement influence coefficients (flexibility coefficients) in the force method equations can be computed by the unit load method (Maxwell-Mohr integral method). Furthermore, the redundant constraint forces can be determined by solving the force method equations. Finally, the remaining support reactions and internal forces of the statically indeterminate structure can be obtained from the static equilibrium condition. For the same structure, there are several ways to choose the primary structure and primary unknowns of the force method, while the number of primary unknowns is identical for different

primary structures. For each redundant constraint force, there is a corresponding compatibility condition of deformation. Since there are various ways to choose the primary structure, the force method is very flexible and full of skillfulness.

In this chapter, the fundamental principle of the force method is introduced, and the calculation of internal forces for various forms of statically indeterminate structures under loadings, support movement, and temperature change is discussed. Meanwhile, the analysis of the force and deformation characteristics of various structures is highlighted.

5.1 Determination of Degree of Static Indeterminacy

5.1.1 *Equilibrium and Geometric Construction Characteristics of Statically Indeterminate Structures*

If the support reactions and internal forces at each section of a structure can be uniquely determined from the static equilibrium condition, it is called a statically determinate structure, such as the simply supported beam shown in figure 5.1a. If the internal forces and support reactions of a structure cannot be directly determined from the static equilibrium condition, it is a **statically indeterminate structure**, such as the **continuous beam** shown in figure 5.1b.

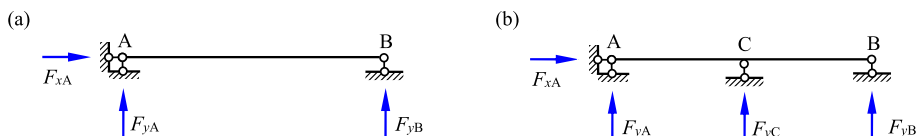


FIG. 5.1 – (a) Statically determinate structure and (b) statically indeterminate structure.

From the perspective of geometric construction, both the simply supported beam and the continuous beam are geometrically stable. If support link B is removed from the simply supported beam, the structure becomes a geometrically unstable system; however, if support link C is removed from the continuous beam, it is still geometrically stable. Therefore, for the continuous beam, support link C is a redundant constraint. This leads to the following conclusions: a statically determinate structure is a geometrically stable system without redundant constraints, while a statically indeterminate structure is a geometrically stable system with redundant constraints. Therefore, there are two main characteristics that distinguish a statically indeterminate structure from a statically determinate structure: the internal forces are statically indeterminate, and there are redundant constraints.

5.1.2 Determination of Degree of Indeterminacy and Number of Redundant Constraint Forces

For a statically indeterminate structure, the number of redundant constraint forces is the same as the number of redundant constraints. To determine the redundant constraint forces, the same number of deformation compatibility equations need to be established. Therefore, to solve a statically indeterminate structure by the force method, the number of redundant constraints must be determined first. This number is called the **degree of indeterminacy** of the structure. A statically indeterminate structure is a geometrically stable system. According to section 2.3, the degree of indeterminacy n is equal to negative computational degree of freedom W :

$$n = -W \quad (a)$$

A statically indeterminate structure can be considered as a statically determinate structure plus several redundant constraints. Therefore, the most direct way of determining the degree of indeterminacy is to remove the redundant constraints from the original structure to make it become a statically determinate structure. Then, the number of removed redundant constraints is the degree of indeterminacy of the original structure. From the perspective of static analysis, the degree of indeterminacy is equal to the number of equations lacked when calculating the unknown forces in terms of the equilibrium equations, that is, the number of redundant constraint forces.

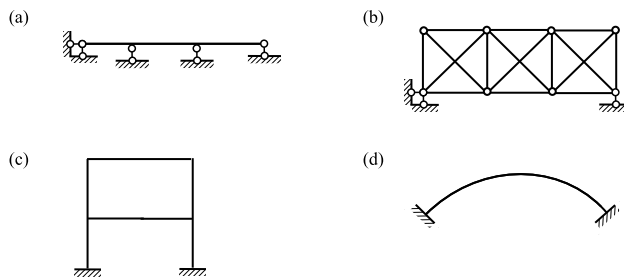


FIG. 5.2 – Various statically indeterminate structures.

For example, after removing or cutting off the extra constraints, the statically indeterminate structures shown in figure 5.2a–d will become the statically determinate structures shown in figure 5.3a–d. The corresponding redundant constraint forces that are exposed after removing the redundant constraints are also plotted in the figure. The degrees of indeterminacy of these structures are 2, 3, 6, and 3, respectively.

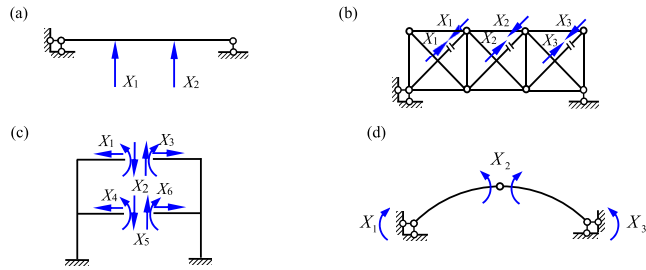


FIG. 5.3 – Statically determinate structures after removing the redundant constraints.

For the case of plane structures, to obtain a statically determinate structure by removing redundant constraints from a statically indeterminate structure, the following points should be noted:

- (1) Removing a support link or cutting off a link is equivalent to removing one constraint (figure 5.3a, b).
- (2) Removing a pin support or a simple hinge is equivalent to removing two constraints.
- (3) Removing a fixed support or cutting off a beam is equivalent to removing three constraints (figure 5.3c).
- (4) Changing a fixed support as a pin support or slider support, adding a simple hinge to a continuous bar, or changing a pin support or slider support as a roller support, are all equivalent to removing one constraint (figure 5.3d).
- (5) The original structure should not be disassembled into a geometrically unstable system, that is, the necessary constraints cannot be removed. For example, if the horizontal support link in the beam shown in figure 5.2a is removed, it becomes a geometrically unstable system.
- (6) All redundant constraints should be removed. For example, for the structure in figure 5.4a, if only one vertical support link is removed, as shown in figure 5.4b, the closed frame still has three redundant constraints. The closed frame must be cut off at a section, as shown in figure 5.4c, to become a statically determinate structure. Therefore, the original structure has a total of four redundant constraints: one from the support, and the other three from the internal structure.

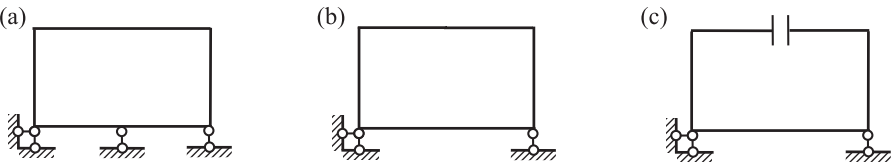


FIG. 5.4 – A closed frame with internal redundant constraints.

5.2 Fundamental Concept of the Force Method

When using the force method to solve the problem of a statically indeterminate structure, the structure is related and compared to a statically determinate structure, so as to find a way to transform the statically indeterminate problem into a statically determinate problem. To analyze a statically indeterminate structure by the force method, the redundant constraint forces are chosen as the primary unknowns, which can be solved according to the deformation compatibility condition. Then, the redundant constraint forces, together with the external loads, are applied to the primary structure. The reactions and internal forces of the structure can be determined according to the static equilibrium conditions. It can be seen that the key to analyzing a statically indeterminate structure by the force method is to establish the compatibility equation, from which the redundant constraint forces can be solved. The compatibility equation is corresponding to the equation of the force method.

5.2.1 Primary Unknowns, Primary System, and Basic Equations of Force Method

Next, we will introduce the three basic concepts of the force method by using the structure with a single degree of indeterminacy shown in figure 5.5a.

1. Primary unknowns

By comparing the statically indeterminate structure shown in figure 5.5a with the statically determinate structure shown in figure 5.5b, we know:

There are three unknown reactions F_{xA} , F_{yA} , and M_A in figure 5.5b, which can be determined by the three equilibrium equations.

In figure 5.5a, there is one redundant unknown constraint force X_1 at support B. The system contains a total of four unknown constraint reactions, which cannot be completely determined from the three equilibrium equations. Therefore, the new problem encountered in solving this statically indeterminate structure is to solve the redundant unknown force X_1 . Once X_1 is determined, the problem becomes a statically determinate problem.

Removing the redundant constraints in the structure and replacing them with the unknown forces, these unknown forces are called the **primary unknowns of force method**. In the force method, the calculation of redundant unknown forces is the key problem for the analysis of statically indeterminate structures, and the redundant constraint forces are the key unknown forces. This is why the method is called the force method.

As long as X_1 is determined, the remaining unknown forces can be solved easily. Therefore, in the force method, the calculation of the primary unknown X_1 is the primary goal. It is worth pointing out that in figure 5.5a, F_{yA} or M_A can also be chosen as the primary unknown, but F_{xA} cannot.

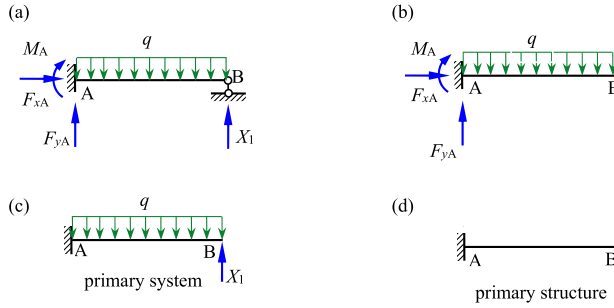


FIG. 5.5 – Primary unknowns, primary system and primary structure of force method.

2. Primary system

By removing the redundant constraint support B in figure 5.5a and replacing it with the redundant unknown force X_1 , we obtain a statically determinate structure with redundant unknown force as shown in figure 5.5c, which is called the **primary system of force method**. If both the redundant constraint (support B) and the external loads are removed from the statically indeterminate structure in figure 5.5a, the obtained statically determinate structure is called the **primary structure of force method** (figure 5.5d).

Since the primary system retains the redundant constraint reaction X_1 in the original structure, the force state of the primary system is exactly the same as the original structure. Therefore, the primary system can represent both the statically determinate structure and the original statically indeterminate structure, *i.e.*, it is the bridge between the statically determinate structure and the statically indeterminate structure.

3. Basic equation

As the primary unknown X_1 in figure 5.5a cannot be determined only by the static equilibrium condition, a new equation must be added. In the statically determinate structure, as shown in figure 5.5c, there will be no vertical displacement in the movable hinged support at point B in the original structure. Therefore, the primary structure must also meet this deformation compatibility condition under the combined action of the external loads and the redundant constraint force. That is, the displacement Δ_1 along the direction of the redundant constraint force X_1 at point B should be zero, and then the primary system can be completely equivalent to the original statically indeterminate structure.

Therefore, the condition of transforming the primary system to the original statically indeterminate structure is: the displacement of the primary system along the direction of the redundant constraint force X_1 should be the same as the original structure, *i.e.*,

$$\Delta_1 = 0 \quad (\text{a})$$

This transforming condition is a compatibility condition of deformation, which is the supplementary condition required for calculating the redundant unknown force.

For a linear deformable system, by using the **principle of superposition**, the deformation condition (a) can be written as an equation containing the redundant unknown force X_1 , which is called the **basic equation of force method**.

Figure 5.6a shows a primary system under the combined action of load q and unknown force X_1 . According to the principle of superposition, state (a) in figure 5.6 is equivalent to the superposition of state (b) and state (c). Here the states (b) and (c) represent the states of the primary structure under the independent action of q and X_1 , respectively, as shown in figure 5.6b and c. Therefore, the deformation compatibility condition (a) can be expressed as:

$$\Delta_1 = \Delta_{1P} + \Delta_{11} = 0 \quad (b)$$

where, Δ_1 is the total displacement of the primary system in the X_1 direction under the combined action of load q and the unknown force X_1 (*i.e.*, the vertical displacement of point B in figure 5.6a).

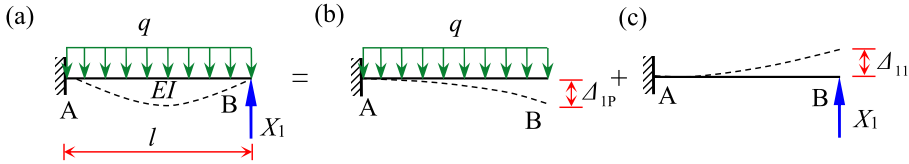


FIG. 5.6 – Deformation compatibility condition of force method.

Δ_{1P} is the displacement of the primary structure along the X_1 direction under load q alone (figure 5.6b). Δ_{11} is the displacement of the primary structure along the X_1 direction under the action of the unknown force X_1 alone (figure 5.6c). If the directions of the displacements Δ_1 , Δ_{1P} and Δ_{11} are the same as the positive direction of the unknown force X_1 , they are defined as positive.

When $X_1 = 1$ acts on the primary structure alone, the displacement of point B along the X_1 direction is denoted as δ_{11} . According to the principle of superposition, when the unknown force X_1 acts on the primary structure, the displacement caused by X_1 can be written as

$$\Delta_{11} = \delta_{11} X_1 \quad (c)$$

Therefore, the physical meaning of the coefficient δ_{11} is the displacement of the primary structure along the X_1 direction under the unit load $X_1 = 1$ (figure 5.7b). Substituting equation (c) into equation (b), we get

$$\delta_{11} X_1 + \Delta_{1P} = 0 \quad (5.1)$$

This is the basic equation of the force method of a structure with a single degree of indeterminacy under linearly elastic deformation.

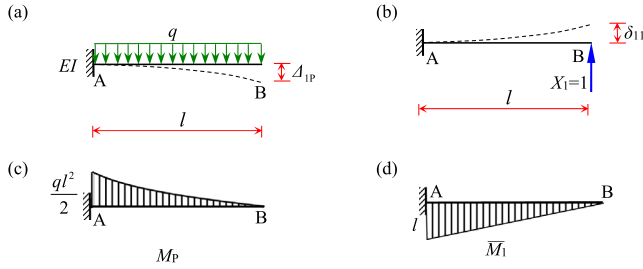


FIG. 5.7 – The coefficient and free term in the basic equation of force method.

The coefficient δ_{11} and free term Δ_{1P} in the basic equation of force method are both displacements of the statically determinate primary structure. To calculate δ_{11} and Δ_{1P} , the moment diagram M_P (figure 5.7c) of the primary structure under external loads and the moment diagram $\bar{M}_1$ (figure 5.7d) under the unit load $X_1 = 1$ should be drawn first. Then, by using the fast integral method of unit load method for structural displacement calculation, *i.e.*, the graph multiplication method, we have

$$\Delta_{1P} = \int \frac{\bar{M}_1 M_P}{EI} dx = -\frac{1}{EI} \times \left(\frac{1}{3} \times \frac{ql^2}{2} \cdot l \right) \times \frac{3l}{4} = -\frac{ql^4}{8EI}$$

$$\delta_{11} = \int \frac{\bar{M}_1 \bar{M}_1}{EI} dx = \frac{1}{EI} \times \left(\frac{l \cdot l}{2} \times \frac{2l}{3} \right) = \frac{l^3}{3EI}$$

Substituting them into the force method equation (5.1) yields

$$\frac{l^3}{3EI} X_1 - \frac{ql^4}{8EI} = 0$$

Then we have

$$X_1 = \frac{3}{8} ql$$

The obtained unknown force is positive, indicating that the direction of X_1 is the same as the originally assumed direction.

After the redundant unknown force is determined, the support reactions of the original structure and the internal forces of each section can be calculated using the equilibrium conditions according to the internal force solution method of statically determined structures. The results are illustrated in figure 5.8a, b, and c.

According to the principle of superposition, the moment M of any section of the structure can be expressed by the following formula:

$$M = \bar{M}_1 X_1 + M_P \quad (5.2)$$

where, $\bar{M}_1$ is the moment in the primary structure caused by the unit load $X_1=1$; M_P is the moment caused by the external loads.

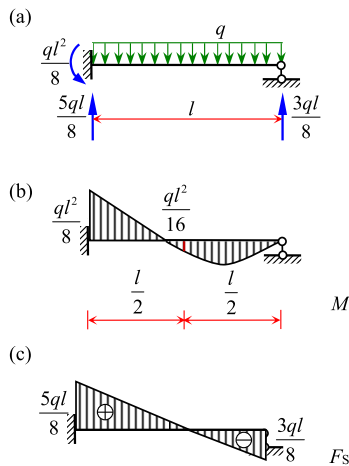


FIG. 5.8 – The support reactions and internal force diagrams of original structure.

5.2.2 Analysis of Structures with Multiple Degrees of Indeterminacy by Force Method

As shown in figure 5.9a, a frame with two degrees of indeterminacy is used to illustrate the main points and process of the analysis of structures with multiple degrees of indeterminacy by the force method. Taking the reactions X_1 and X_2 at support B as the primary unknowns, the primary system is shown in figure 5.9b, and the corresponding primary structure is shown in figure 5.9c.

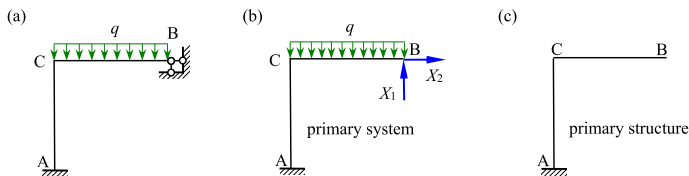


FIG. 5.9 – Primary system and primary structure of a frame with two degrees of indeterminacy.

The deformation compatibility condition of the redundant constraints is used to determine the redundant unknown forces X_1 and X_2 . The deformation compatibility condition is: the displacements at point B of the primary system along the X_1 and X_2 directions should be equal to those of the original structure, *i.e.*, be zero. Thus we have

$$\left. \begin{aligned} \Delta_1 &= 0 \\ \Delta_2 &= 0 \end{aligned} \right\} \quad (d)$$

where, Δ_1 is the displacement of the primary system in the X_1 direction, *i.e.*, the vertical displacement of point B; Δ_2 is the displacement of the primary system in the X_2 direction, *i.e.*, the horizontal displacement of point B.

The deformation compatibility equation (d) can be expanded using the principle of superposition. First, the displacements of the primary structure under the action of external loads, the unknown force X_1 and the unknown force X_2 are calculated separately:

- (1) When the external loads are applied alone, the vertical and horizontal displacements of point B are Δ_{1P} and Δ_{2P} , respectively (figure 5.10a).
- (2) When unit load $X_1 = 1$ is applied alone, the vertical and horizontal displacements of point B are δ_{11} and δ_{21} , respectively (figure 5.10b); when unknown force X_1 is applied alone, the vertical and horizontal displacements of point B are $\delta_{11}X_1$ and $\delta_{21}X_1$, respectively.
- (3) When unit load $X_2 = 1$ is applied alone, the vertical and horizontal displacements of point B are δ_{12} and δ_{22} , respectively (figure 5.10b); when unknown force X_2 is applied alone, the vertical and horizontal displacements of point B are $\delta_{12}X_2$ and $\delta_{22}X_2$, respectively.

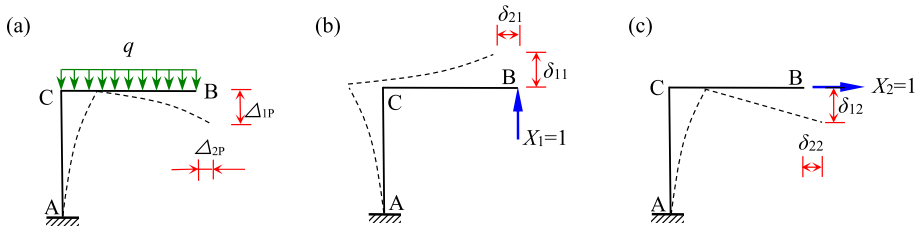


FIG. 5.10 – The coefficients and free terms for force method analysis of frame.

Then, according to the principle of superposition, the total displacements of point B in the vertical and horizontal directions under the combined action of external loads and unknown forces are

$$\Delta_1 = \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P}$$

$$\Delta_2 = \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P}$$

Therefore, the deformation compatibility condition (d) becomes

$$\left. \begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P} &= 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P} &= 0 \end{aligned} \right\} \quad (5.3)$$

The above formulas are the basic equations of force method for the structures with two degrees of indeterminacy.

After the redundant unknown forces X_1 and X_2 are determined from the basic equations, the support reactions and internal forces of the original structure can be

solved by using the equilibrium conditions, according to the internal force calculation method of statically determinate structures. In addition, the internal forces of the primary structure caused by external loads, the unknown forces X_1 and X_2 alone can be calculated separately, and then the internal forces of the original structure can be solved using the principle of superposition. For example, the moment M of any section can be written as:

$$M = \overline{M}_1 X_1 + \overline{M}_2 X_2 + M_P$$

where, M_P is the moment of the primary structure caused by external loads; $\overline{M}_1$ and $\overline{M}_2$ are the moments of the primary structure caused by unit load $X_1 = 1$ and $X_2 = 1$, respectively.

There are different ways to select the primary system and primary unknowns of the force method for the same structure, but the final results should be the same. For example, for the structure shown in figure 5.9a, the primary system can also be selected as the system shown in figure 5.11a or b. In these cases, the form of the basic equation of force method is the same as equation (5.3), but the meaning of the corresponding deformation condition is different, as the meaning of X_1 and X_2 is different for different primary systems. It is worth noting that the primary system should be geometrically stable. Therefore, the instantaneously unstable system shown in figure 5.11c cannot be chosen as a primary system.

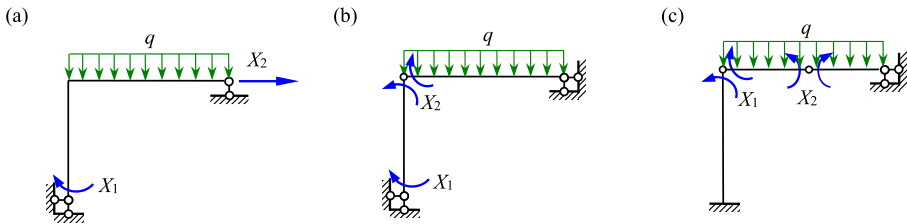


FIG. 5.11 – Different selections of primary system and primary unknowns of force method.

5.2.3 Canonical Equations of Force Method

For the statically indeterminate structures with n degrees of indeterminacy, there are n redundant constraints, each of which corresponds to a primary unknown—the unknown restraint force and each redundant restraint provides a deformation condition. Thus, n deformation compatibility equations can be established for the primary system accordingly, namely the basic equations of the force method, from which the n unknown restraint forces can be solved. These n equations can be written as:

$$\left. \begin{aligned} \delta_{11} X_1 + \delta_{12} X_2 + \cdots + \delta_{1n} X_n + \Delta_{1P} &= \Delta_1 \\ \delta_{21} X_1 + \delta_{22} X_2 + \cdots + \delta_{2n} X_n + \Delta_{2P} &= \Delta_2 \\ &\dots\dots\dots \\ \delta_{n1} X_1 + \delta_{n2} X_2 + \cdots + \delta_{nn} X_n + \Delta_{nP} &= \Delta_n \end{aligned} \right\} \quad (5.4a)$$

If the true displacements of the original structure at the redundant constraints are all zero, then we have

$$\left. \begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \cdots + \delta_{1n}X_n + \Delta_{1P} &= 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \cdots + \delta_{2n}X_n + \Delta_{2P} &= 0 \\ &\dots\dots\dots \\ \delta_{n1}X_1 + \delta_{n2}X_2 + \cdots + \delta_{nn}X_n + \Delta_{nP} &= 0 \end{aligned} \right\} \quad (5.4b)$$

Equation (5.4a) or (5.4b) is the general form of the basic equations of force method for statically indeterminate structures with n degrees of indeterminacy under loads. Regardless of the structure type and how the primary structure is selected, the form of the force method equation is constant. Therefore, equation (5.4a) or (5.4b) is often called the **canonical equations of force method**, whose essence is a set of compatibility equations of deformation.

In the canonical equations of force method, the coefficient δ_{ij} is the displacement of the primary structure along the X_i direction caused by the unit load $X_j = 1$. The coefficient δ_{ij} is often referred to as the **flexibility coefficient**, *i.e.*, the displacement influence coefficient introduced in section 4.7. The free term Δ_{iP} represents the displacement of the primary structure in the X_i direction caused by external loads. The displacement is positive when its direction agrees with the primary unknown, otherwise negative. The first letter in this double-subscript notation indicates the direction of the displacement along the corresponding redundant unknown force, and the second one refers to the cause of the resulting displacement. The coefficients δ_{ii} on the main diagonal from the upper left δ_{11} to the lower right δ_{nn} in the force method equation are called the primary coefficients, and the other coefficients $\delta_{ij}(i \neq j)$ are called secondary coefficients. The primary coefficients are always positive, while the secondary coefficients may be positive, negative, or zero. According to the theorem of reciprocal displacements, we have

$$\delta_{ij} = \delta_{ji} \quad (5.5)$$

The canonical equations of force method (5.4a) or (5.4b) can also be written as the following matrix form:

$$\delta \mathbf{X} + \Delta_P = \Delta \quad (5.6a)$$

or

$$\delta \mathbf{X} + \Delta_P = \mathbf{0} \quad (5.6b)$$

where, δ is called the **flexibility matrix**, whose elements are the flexibility coefficients δ_{ij} in equation (5.4a) or (5.4b). From equation (5.5), we can see that δ is a symmetric matrix. The quadratic form is constituted by the flexibility matrix and the non-zero force vector represents the strain complementary energy of the structure, which is constantly positive, thus δ is a positive definite matrix; $\mathbf{X}$ is the vector of primary unknown forces; Δ_P is the displacement vector caused by external loads. The canonical equations of force method represent the compatibility conditions of deformation, which are also called the structural flexibility equations. Therefore, the

force method is also called the **flexibility method**. The canonical equations of the force method can be deduced by using the principle of minimum complementary energy.

The force method equations are a set of linear algebraic equations. All the primary unknowns (the redundant constraint forces) can be obtained by solving the equations. Then, the internal forces of structure can be determined from the equilibrium conditions, or they can be calculated more conveniently from equation (5.7) according to the principle of superposition:

$$\left. \begin{aligned} M &= \overline{M}_1 X_1 + \overline{M}_2 X_2 + \cdots + \overline{M}_n X_n + M_P \\ F_S &= \overline{F}_{S1} X_1 + \overline{F}_{S2} X_2 + \cdots + \overline{F}_{Sn} X_n + F_{SP} \\ F_N &= \overline{F}_{N1} X_1 + \overline{F}_{N2} X_2 + \cdots + \overline{F}_{Nn} X_n + F_{NP} \end{aligned} \right\} \quad (5.7)$$

where, $\overline{M}_i$, $\overline{F}_{Si}$ and $\overline{F}_{Ni}$ are the internal forces of the primary structure under the action of $X_i = 1$; M_P , F_{SP} , and F_{NP} are the internal forces of the primary structure under external loads. After the moment of the original structure is obtained from the first formula of equation (5.7), F_S and F_N can be directly solved from the equilibrium conditions.

5.3 Analysis of Statically Indeterminate Frames and Bent Structures

According to the fundamental principles of the force method, the analysis of a statically indeterminate structure can be performed by the following steps:

- (1) Determine the degree of indeterminacy of the structure, select a reasonable primary structure, and then apply the external loads and redundant constraint forces (the primary unknowns of the force method) to the primary structure.
- (2) Establish the force method equations, and determine the flexibility coefficients δ_{ij} and free terms Δ_{iP} . To do this, the unit internal force diagram (under the action of each unit unknown force alone) and load internal force diagram (under the external load) of the primary structure need to be drawn first, and then the flexibility coefficients and free terms can be calculated according to the displacement calculation method of the statically determinate structures.
- (3) Solve the force method equations to obtain the primary unknowns, *i.e.*, the redundant constraint forces.
- (4) Use the principle of superposition to draw the internal force diagram of the primary structure under the combined action of external load and redundant constraint forces, which is actually the internal force diagram of the original structure.

If a different primary structure is selected, the calculation step and final results of the statically indeterminate structure should be the same, but the calculation workload may be quite different. Therefore, the selection of a proper primary structure is of great significance in the force method. The general principle for the

selection of the primary structure is to make the calculation simple. For example, for beam and frame structures, the unit moment diagram and external load moment diagram should be simple, or some secondary coefficients or free terms of the equations should be zero, so as to facilitate the application of the graph multiplication method. If the structure is symmetric, generally a symmetric primary structure should be selected. For the case of a structure with elastic supports, the elastic supports can be cut off to remove the redundant constraints, making the calculation relatively simple.

For the members of frames and bent structures, generally, the influence of axial and shear force on deformation is much smaller than the moment. Therefore, in the analysis of statically indeterminate frames and bent structures, the influence of axial and shear deformation can usually be ignored. However, for the columns of high-rise frames, the influence of axial force is relatively large; while for stubby members, the influence of shear force is relatively large. Under these situations, special treatment is required.

As shown in figure 5.12 is the computational model of a bent structure of a single-story single-span plant. The columns are members with stepped variable cross-sections. The bottom of the columns is fixed, and the top of the columns is hinged with a crossbeam (roof truss). The axial deformation of the crossbeam is often ignored in the calculation.



FIG. 5.12 – A single-story single-span plant bent.

Example 5.1. As shown in figure 5.13a is a statically indeterminate frame. The inertia moments of the beam and column are I_1 and I_2 , and $I_1:I_2 = 2:1$. The beam is subjected to a uniformly distributed load $q = 20 \text{ kN/m}$. Draw the internal force diagram of the frame.

Solution: (1) Select the primary system

The frame has one degree of indeterminacy. Select the horizontal reaction at point B as the redundant unknown force. After removing the horizontal support at point B and replacing it with an unknown force X_1 , the primary system shown in figure 5.13b can be obtained.

(2) Establish the force method equation

There is no horizontal displacement at point B in the original structure. From this deformation condition, the force method equation can be established as:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

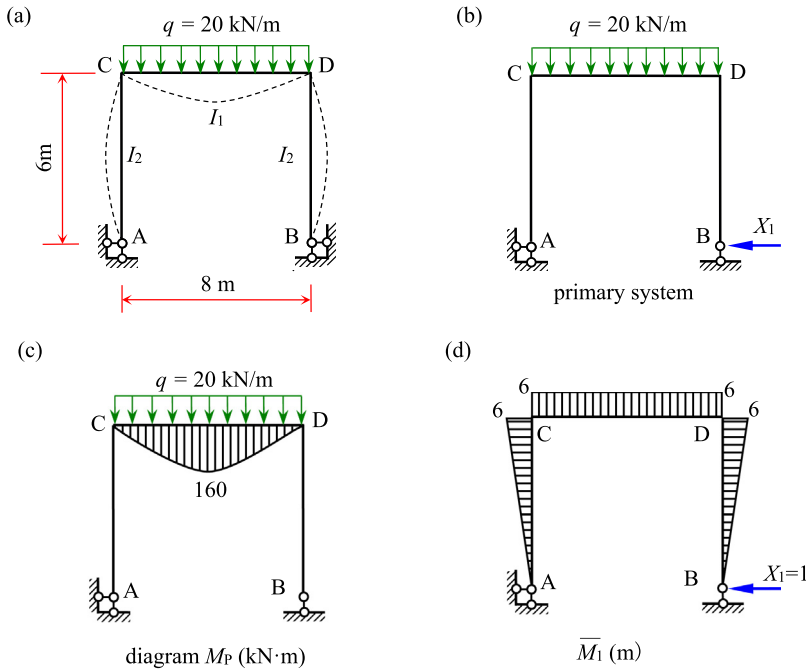


FIG. 5.13 – Analysis of a statically indeterminate frame by force method.

(3) Determine the coefficients and free term

Both the coefficient δ_{11} and free term Δ_{1P} are displacements of the statically determinate primary structure. In calculation, the axial and shear forces can be ignored, and only the influence of the moment on the displacement of the frame is considered. Draw the moment diagram of the primary structure under external loads (namely, the load moment diagram M_P), and the moment diagram under unit load $X_1 = 1$ (the unit moment diagram $\bar{M}_1$), respectively, as shown in figure 5.13c and d.

Calculate the displacement by the graph multiplication method:

$$\begin{aligned}\Delta_{1P} &= \sum \int \frac{\bar{M}_1 M_P}{EI} dx = -\frac{1}{EI_1} \times \left(\frac{2}{3} \times 160 \text{ kN} \cdot \text{m} \times 8 \text{ m} \right) \times 6 \text{ m} \\ &= -\frac{5120 \text{ kN} \cdot \text{m}^3}{EI_1}\end{aligned}$$

Here, as the diagrams M_P and $\bar{M}_1$ are on the opposite side of the beam, the product of the two diagrams is negative.

$$\begin{aligned}\delta_{11} &= \sum \int \frac{\bar{M}_1 \bar{M}_1}{EI} dx \\ &= \frac{1}{EI_1} \times (6 \text{ m} \times 8 \text{ m}) \times 6 \text{ m} + \frac{2}{EI_2} \times \left(\frac{1}{2} \times 6 \text{ m} \times 6 \text{ m} \right) \times \left(\frac{2}{3} \times 6 \text{ m} \right) \\ &= \frac{288 \text{ m}^3}{EI_1} + \frac{144 \text{ m}^3}{EI_2}\end{aligned}$$

As $I_1:I_2 = 2:1$, we have

$$\delta_{11} = \frac{576 \text{ m}^3}{EI_1}$$

(4) Solve the redundant unknown force

Substituting δ_{11} and Δ_{1P} into the force method equation, we get

$$\frac{576 \text{ m}^3}{EI_1} X_1 - \frac{5120 \text{ kN} \cdot \text{m}^3}{EI_1} = 0$$

Thus,

$$X_1 = \frac{80}{9} \text{ kN}$$

The calculation result is positive, indicating that the actual direction of the primary unknown force is the same as the assumed direction; otherwise opposite. Since all the terms in the force equation contain EI_1 , it can be eliminated. Therefore, when calculating the internal forces of a statically indeterminate frame under loadings, we only need to know the relative value of EI of each member, instead of the absolute value. In other words, the internal and reaction forces of a statically indeterminate structure under loadings are irrelevant to the absolute stiffness of the members. When the stiffness ratio of each member changes, the statically indeterminate forces will also change. As long as the relative value of the stiffness of each member remains unchanged, the internal forces are unrelated to the material and cross-sectional dimension of the structure. It is worth noting that this conclusion is only applicable to the case of loading. The internal forces of statically indeterminate structures caused by support movement or temperature change are dependent on the absolute value of the member stiffness.

(5) Draw the internal force diagram

After the redundant unknown force is obtained, the internal forces of the structure can be determined from the equilibrium condition, which is a problem of statically determinate structure. Generally, the step of drawing the internal force diagram is: first, draw the resultant moment diagram from the diagrams $\overline{M}_1$ and M_P ; then, draw the shear diagram according to the moment diagram; finally, draw the axial force diagram according to the shear diagram. The process is described as follows:

1) Moment diagram

The superposition formula of moment is

$$M = \overline{M}_1 X_1 + M_P$$

Multiplying the diagram $\overline{M}_1$ by $X_1 = 80/9 \text{ kN}$ and then adding the diagram M_P , we get diagram M (figure 5.14a).

2) Shear diagram

To draw the shear diagram of a member, the member can be regarded as a free body. The shears at the member ends can be calculated by the equilibrium conditions from the known moments, and then the shear diagram can be drawn.

Taking member CD as an example, the free body is shown in figure 5.14b (the axial forces at the member ends are not drawn as they have no contribution to the section moment). After obtaining the shears at the member ends, the diagram F_S of member CD can be drawn, as shown in figure 5.14c.

3) Axial force diagram

To draw the axial force diagram of a member, the joint can be taken as the free body. The axial forces at the member ends can be calculated by the equilibrium conditions from the known shears, and then the axial force diagram can be drawn.

Taking joint C as an example, the free body is shown in figure 5.14b (the moments acting on the free body are not drawn). Each joint has two equilibrium equations. By analyzing the joints in a proper order, all the member end axial forces can be determined. The axial force diagram is shown in figure 5.14d.

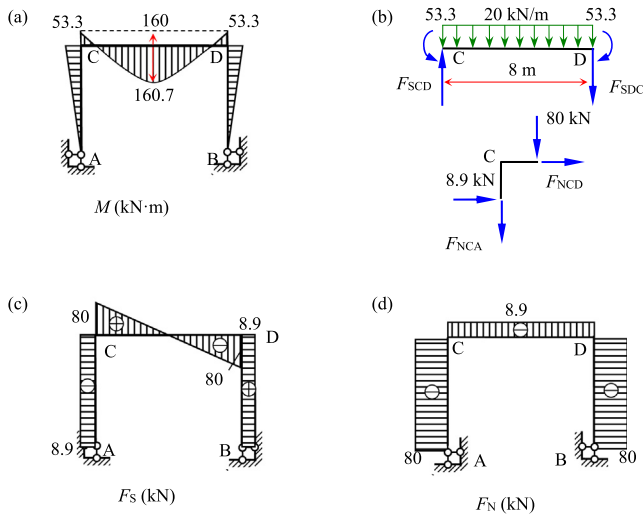


FIG. 5.14 – Internal force diagrams of the frame.

The deformation of the statically indeterminate frame is shown by the dashed line in figure 5.13a. From the moment diagram of the frame (figure 5.14a), it can be seen that the outside of the two vertical columns is stretched, so the deformation curve should be outward convex. For the beam, the outside of the cross-section at both ends is stretched, so the deformation is also outward convex; The inside of the middle section is stretched, so the deformation is inward convex. It can be seen that the deformation mode derived from the moment diagram of the frame is consistent with the actual deformation curve. In turn, the shape of the moment diagram can

also be estimated from the deformation curve of the frame in combination with the load style. As shown in this example, a qualitative analysis of the force and deformation behavior of a statically indeterminate structure is helpful for quick judgment of the correctness of the internal force diagram and for the conceptual design of engineering structures.

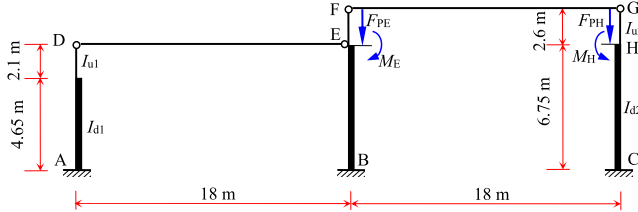


FIG. 5.15 – Computational model of a two-span plant bent.

Example 5.2. As shown in figure 5.15 is the computational model of a two-span plant bent. Determine the internal forces under the crane loads. The calculation conditions are as follows:

(1) Inertia moment of sections

Left column: upper section $I_{u1} = 10.1 \times 10^4 \text{ cm}^4$, lower section $I_{d1} = 28.6 \times 10^4 \text{ cm}^4$.

Right and middle columns: upper section $I_{u2} = 16.1 \times 10^4 \text{ cm}^4$, lower section $I_{d2} = 81.8 \times 10^4 \text{ cm}^4$.

(2) Right-span crane loads

The vertical loads are $F_{PH} = 108 \text{ kN}$ and $F_{PE} = 43.9 \text{ kN}$. As there is an eccentric distance of $e = 0.4 \text{ m}$ between F_{PH} , F_{PE} , and the axis of the lower column, the moments at points H and E are

$$M_H = F_{PH} \cdot e = 43.2 \text{ kN} \cdot \text{m}, \quad M_E = F_{PE} \cdot e = 17.6 \text{ kN} \cdot \text{m}$$

Solution: The bent structure has two degrees of indeterminacy. Members FG and DE are hinged at both ends, which act as links under the crane loads that are only subjected to axial force. The crane loads cause not only the couple moments M_E and M_H at sections E and H but also vertical pressure. Since the vertical pressure causes only axial force in the lower columns, the vertical forces F_{PE} and F_{PH} are not drawn in the primary system in figure 5.16a when calculating the internal moments.

Take the axial forces X_1 and X_2 of links FG and DE as the redundant unknown forces. By cutting off the axial constraints of the two links and adding the axial forces X_1 and X_2 , the primary system is obtained as shown in figure 5.16a. It is worth noting that the redundant unknown forces X_1 and X_2 are all internal forces. They are the generalized forces in the primary system, composed of a pair of forces with equal values and opposite directions. In addition, cutting off a member usually means removing all three constraints, *i.e.*, the axial force, shear, and moment. In this example, only the axial constraint (the constraint corresponding to the axial force) is

removed. Figure 5.16b shows the notch of member FG. Only the axial constraint is cut off, thus only a pair of axial internal forces are exposed.

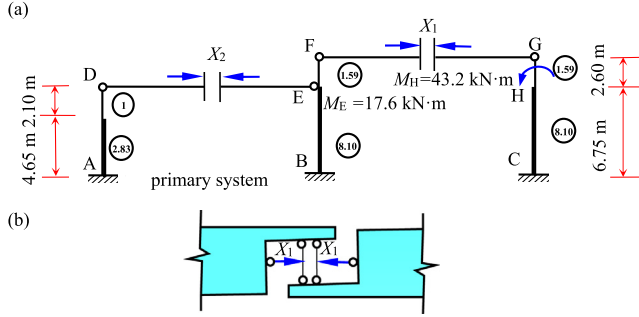


FIG. 5.16 – (a) Primary system of the bent and (b) notch of member FG.

The basic equations of the force method are

$$\Delta_1 = \delta_{11} X_1 + \delta_{12} X_2 + \Delta_{1P} = 0$$

$$\Delta_2 = \delta_{21} X_1 + \delta_{22} X_2 + \Delta_{2P} = 0$$

where, Δ_1 and Δ_2 respectively represent the generalized displacements corresponding to the axial forces X_1 and X_2 , *i.e.*, the relative axial displacements of the two sections at the notch. Therefore, the deformation compatibility conditions represented by the basic equations of the force method are: the two sections at the notch should remain in contact in the axial direction, that is, the relative axial displacement should be zero.

After drawing the diagrams M_P , $\overline{M}_1$ and $\overline{M}_2$ of the primary structure (figure 5.17a, b, c), the free terms and coefficients can be solved as follows (the number enclosed by a circle in figure 5.16a is the relative value of EI of each member, and the resulting displacement is also a relative value):

$$\Delta_{1P} = \sum \int \frac{\overline{M}_1 M_P}{EI} ds = \frac{1}{8.10} \times \frac{2.60 + 9.35}{2} \times 6.75 \times (43.2 + 17.6) = 303$$

$$\Delta_{2P} = \sum \int \frac{\overline{M}_2 M_P}{EI} ds = -\frac{1}{8.10} \times \frac{6.75 \times 6.75}{2} \times 17.6 = -49.5$$

$$\begin{aligned} \delta_{11} = \sum \int \frac{\overline{M}_1^2}{EI} ds &= \frac{1}{1.59} \times \frac{2.6 \times 2.6}{2} \times \frac{2}{3} \times 2.6 \times 2 \\ &+ \frac{1}{8.10} \times \left(2.6 \times 6.75 \times 5.98 + \frac{6.75 \times 6.75}{2} \times 7.10 \right) \times 2 = 73.4 \end{aligned}$$

$$\delta_{22} = \sum \int \frac{\bar{M}_2^2}{EI} ds = \frac{1}{8.10} \times \frac{6.75 \times 6.75}{2} \times \frac{2}{3} \times 6.75 + \frac{1}{1} \times \frac{2.1 \times 2.1}{2} \times \frac{2}{3} \times 2.1$$

$$+ \frac{1}{2.83} \times \left(2.1 \times 4.65 \times 4.43 + \frac{4.65 \times 4.65}{2} \times 5.20 \right) = 50.9$$

$$\delta_{12} = \delta_{21} = -\frac{1}{8.10} \times \frac{6.75 \times 6.75}{2} \times 7.10 = -20$$

The obtained force method equations are

$$\begin{cases} 73.4X_1 - 20X_2 + 303 = 0 \\ -20X_1 + 50.9X_2 - 49.5 = 0 \end{cases}$$

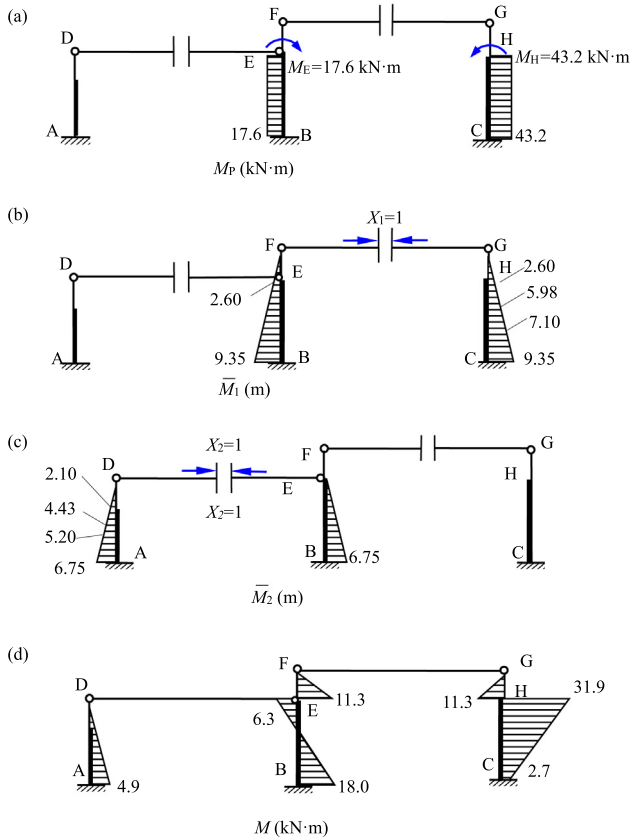


FIG. 5.17 – Analysis of the bent by force method.

Solving the equations yields

$$X_1 = -4.33 \text{ kN}, X_2 = -0.73 \text{ kN}$$

After obtaining the redundant constraint forces of the bent, the diagram M of the structure can be drawn according to the superposition formula $M = \overline{M}_1 X_1 + \overline{M}_2 X_2 + M_P$, as shown in figure 5.17d.

5.4 Analysis of Statically Indeterminate Trusses and Composite Structures

For an ideal truss, the bars can only bear axial force. To calculate the displacement of the truss, we only need to consider the influence of axial deformation of bars. A composite structure consists of both links and beams, which can save materials. Generally, a beam in a composite structure bears not only moments but also shear and axial forces. When calculating with the force method, the influence of axial and shear deformation of the beam can be ignored.

Example 5.3. Determine the internal forces of the statically indeterminate truss shown in figure 5.18a. The cross-sectional areas of bars are shown in table 5.1.

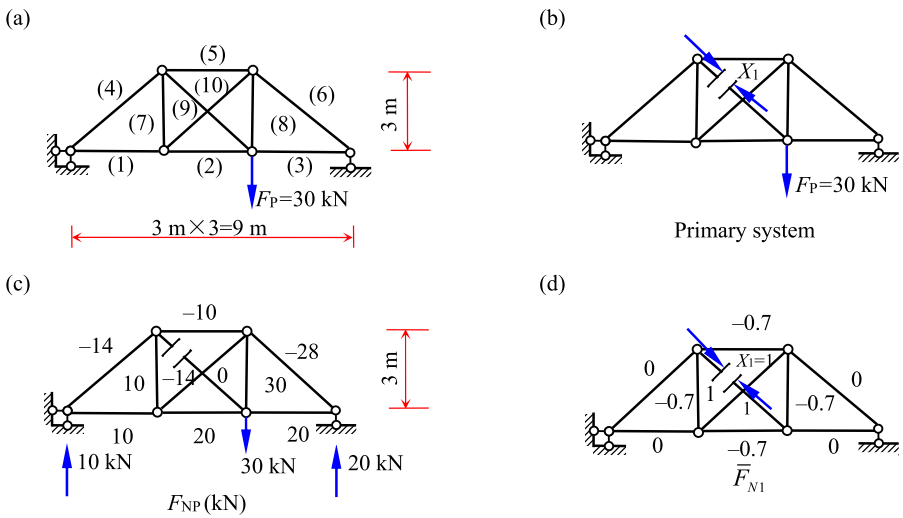


FIG. 5.18 – Analysis of a statically indeterminate truss by force method.

Solution: The truss has a single degree of indeterminacy with one redundant constraint. The primary structure can be selected as the structure after cutting off an inclined bar. The primary system is shown in figure 5.18b. According to the

TAB. 5.1 – Calculation of δ_{11} , A_{1P} and F_N for a statically indeterminate truss.

Bar	l/cm	A/cm^2	F_{NP}/kN	$\overline{F}_{\text{N1}}$	$\frac{\overline{F}_{\text{N1}}^2 l}{A}/\text{cm}^{-1}$	$\frac{\overline{F}_{\text{N1}} F_{\text{NP}} l}{A}/(\text{kN} \times \text{cm}^{-1})$	$F_N = (\overline{F}_{\text{N1}} X_1 + F_{\text{NP}})/\text{kN}$
1	300	15	10	0	0	0	10.0
2	300	20	20	-0.7	7.4	-210	11.5
3	300	15	20	0	0	0	-20.0
4	424	20	-14	0	0	0	-14.0
5	300	25	-10	-0.7	6	84	-18.5
6	424	20	-28	0	0	0	-28.0
7	300	15	10	-0.7	10	-140	1.5
8	300	15	30	-0.7	10	-420	21.5
9	424	15	-14	1	28	-396	-1.9
10	424	15	0	1	28	0	12.1
Σ					89.4	-1082	

deformation compatibility condition (the relative displacement at the notch of the bar should be zero), the force method equation can be established:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

The axial forces $\bar{F}_{N1}$ and F_{NP} of each bar of the primary structure under the action of the unit axial force and the external load alone are calculated separately, as shown in figure 5.18c and d. The calculation formulas for the flexibility coefficient and free term of the force method equation are

$$\delta_{11} = \sum \frac{\bar{F}_{N1}^2 l}{EA}, \Delta_{1P} = \sum \frac{\bar{F}_{N1} F_{NP} l}{EA}$$

The calculation can be performed by using a table, as shown in table 5.1. After substituting the flexibility coefficient and free term into the force method equation, we get

$$X_1 = -\frac{\Delta_{1P}}{\delta_{11}} = -\frac{-1082 \text{ kN/cm} \cdot E}{89.4 \text{ cm}^{-1} \cdot E} = 12.1 \text{ kN}$$

The axial force of each bar can be calculated by the following formula:

$$F_N = \bar{F}_{N1}X_1 + F_{NP}$$

The calculation results are listed in table 5.1.

Example 5.4. Determine the internal forces of the composite structure shown in figure 5.19a under the external loads. The stiffness of each member is given as follows:

AD is a beam, $EI = 1.40 \times 10^4 \text{ kN} \cdot \text{m}^2$, $EA = 1.99 \times 10^6 \text{ kN}$

AC and CD are links, $EA = 2.56 \times 10^5 \text{ kN}$

BC is a link, $EA = 2.02 \times 10^5 \text{ kN}$

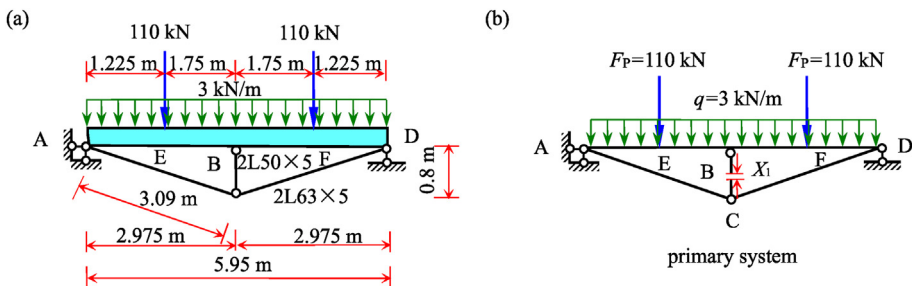


FIG. 5.19 – (a) A statically indeterminate composite structure and (b) primary system.

Solution: (1) Establish primary system and force method equation

This composite structure with stiffened crane beam has a single degree of indeterminacy. By cutting off the redundant link BC, and substituting a pair of

unknown axial forces X_1 of equal magnitude and opposite direction at the notch, we get the primary system shown in figure 5.19b. The relative displacement of the two sections at the notch of the primary system should be zero. So, the force method equation is:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

(2) Determine the coefficient and free term

Apply a unit force $X_1 = 1$ at the notch of the primary structure. The axial force of each member can be obtained by the joint method, as shown in figure 5.20a. Member AD is a beam with a bending moment, and diagram $\bar{M}_1$ is shown in figure 5.20c.

Under the external loads, there are no axial forces in all members of the primary structure, and only beam AD has a moment. The two moments diagrams under the action of the concentrated forces and uniformly distributed load alone are shown in figure 5.20b and d.

$$\begin{aligned}\delta_{11} &= \int \frac{\bar{M}_1^2}{EI} ds + \sum \frac{\bar{F}_{N1}^2 l}{EA} \\ &= \frac{1}{1.4 \times 10^4 \text{ kN} \cdot \text{m}^2} \times \left[\frac{1.49 \text{ m} \times 2.975 \text{ m}}{2} \times \left(\frac{2}{3} \times 1.49 \text{ m} \right) \right] \times 2 \\ &\quad + \frac{1}{1.99 \times 10^6 \text{ kN}} \times (1.86^2 \times 5.95 \text{ m}) \\ &\quad + \frac{1}{2.56 \times 10^5 \text{ kN}} \times (1.93^2 \times 3.09 \text{ m}) \times 2 + \frac{1}{2.02 \times 10^5 \text{ kN}} \\ &\quad \times (1^2 \times 0.80 \text{ m}) = 0.000419 \text{ m/kN}\end{aligned}$$

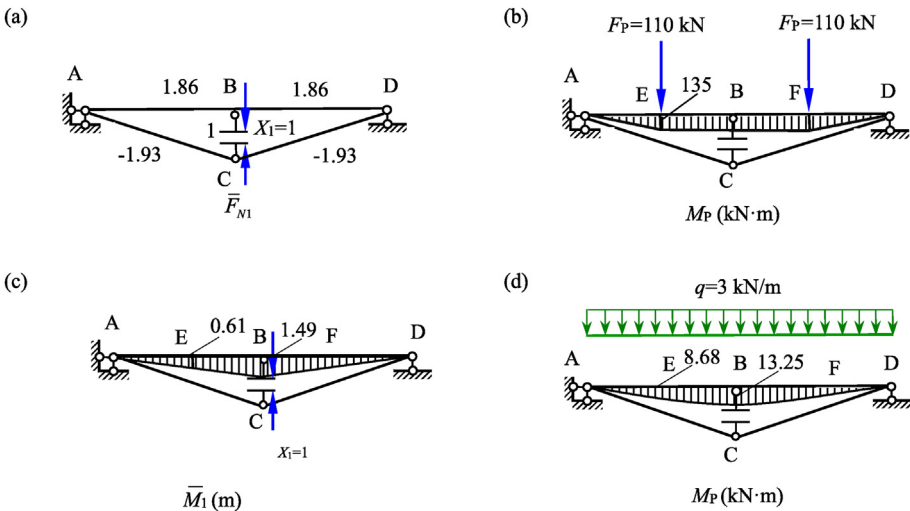


FIG. 5.20 – Analysis of the composite structure by force method.

$$\begin{aligned}
\Delta_{1P} &= \int \frac{\overline{M}_1 M_P}{EI} ds = \frac{1}{1.4 \times 10^4 \text{ kN}\cdot\text{m}^2} \\
&\times \left[\left(\frac{2}{3} \times 13.25 \text{ kN}\cdot\text{m} \times 2.795 \text{ m} \right) \times \left(\frac{5}{8} \times 1.49 \text{ m} \right) \times 2 \right. \\
&+ \left(\frac{1}{2} \times 135 \text{ kN}\cdot\text{m} \times 1.225 \text{ m} \right) \times \left(\frac{2}{3} \times 0.61 \text{ m} \right) \times 2 \\
&\left. + (135 \text{ kN}\cdot\text{m} \times 1.75 \text{ m}) \times \left(\frac{0.61 \text{ m} + 1.49 \text{ m}}{2} \right) \times 2 \right] = 0.0438 \text{ m}
\end{aligned}$$

(3) Solve the redundant unknown force

$$X_1 = -\frac{\Delta_{1P}}{\delta_{11}} = -\frac{0.0438 \text{ m}}{0.000419 \text{ m/kN}} = -104.5 \text{ kN (compression)}$$

(4) Determine the internal forces

$$F_N = \overline{F}_{N1} X_1 + F_{NP}$$

$$M = \overline{M}_1 X_1 + M_P$$

The axial forces of all members and the moment of beam AD are shown in figure 5.21a and b.

(5) Discussion

From the moment of the stiffened beam shown in figure 5.21b, it can be seen that beam AD is subjected to a support reaction of 104.5 kN from the lower truss at midpoint B, with the maximum moment of 79.9 kN·m. If there is no support from the truss, then AD is a simply supported beam. In this case, the moment diagram is shown in figure 5.22a, and the maximum moment is 148.3 kN·m. It can be seen that due to the support of the truss, the maximum moment of the beam is reduced by 46%. However, it should be noted in engineering design that a negative moment appears in the beam after the lower truss is installed, and the shear in the beam between the point of concentrated loading and point B increases significantly. In addition, the vertical bar may be damaged due to its instability under compression. These issues should be fully considered in the cross-sectional and structural design of the beam and the vertical bar. In fact, the rationality of a structure also depends on many factors other than mechanical performance. When solving one aspect of the problem, we must pay attention to other issues caused by it.

It should also be pointed out that the internal force distribution of this statically indeterminate structure is related to the relative stiffness of the beam and the truss. If the cross-sectional area of the lower links is small, the moment diagram M of the beam is close to that of a simply supported beam (figure 5.22a). If the cross-sectional area of the lower links is large, the diagram M of the beam is close to that of a two-span continuous beam (figure 5.22b).

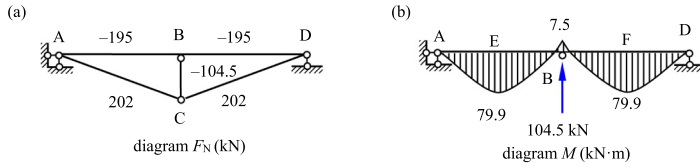


FIG. 5.21 – Internal force diagrams of the composite structure.

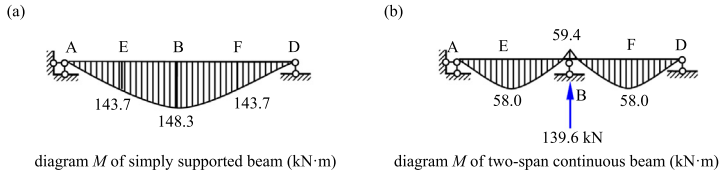


FIG. 5.22 – Effect of the lower truss on the crossbeam.

5.5 Analysis of Symmetric Structures and Half Structures

In actual engineering, many structures are symmetric. The use of symmetry can often simplify the structural analysis. As shown in figure 5.23a is a symmetric single-span frame with one axis of symmetry. The so-called **symmetric structure** refers to:

- (1) The geometric form and support conditions of the structure are symmetric to a specific axis.
- (2) The sectional and material properties of the members are also symmetric about this axis (hence the sectional stiffness EI of the members is symmetric about this axis).

Symmetric structures have the following basic mechanical characteristics: under the action of **symmetric load**, the deformation and internal forces of the structure are all symmetric; under the action of **antisymmetric load**, the deformation and internal forces of the structure are all antisymmetric. A symmetric load means the point of action, magnitude and direction of the load on both sides of the symmetric axis are overlapped after folded about the axis; an antisymmetric load means that the point of action and magnitude of the load on both sides of the symmetric axis are overlapped after folded about the axis, but their directions are inverse. The symmetry or antisymmetry of structural deformation and internal force is also defined according to the above principles. Figure 5.23 illustrates the deformation and moment diagrams of a symmetric frame under symmetric and antisymmetric loads.

Any loads acting on a symmetric structure can be decomposed into a set of symmetric loads and another set of antisymmetric loads. For example, as shown in figure 5.24 is a symmetric frame subjected to a concentrated load and triangular distributed load, which can be decomposed into symmetric and antisymmetric loads.

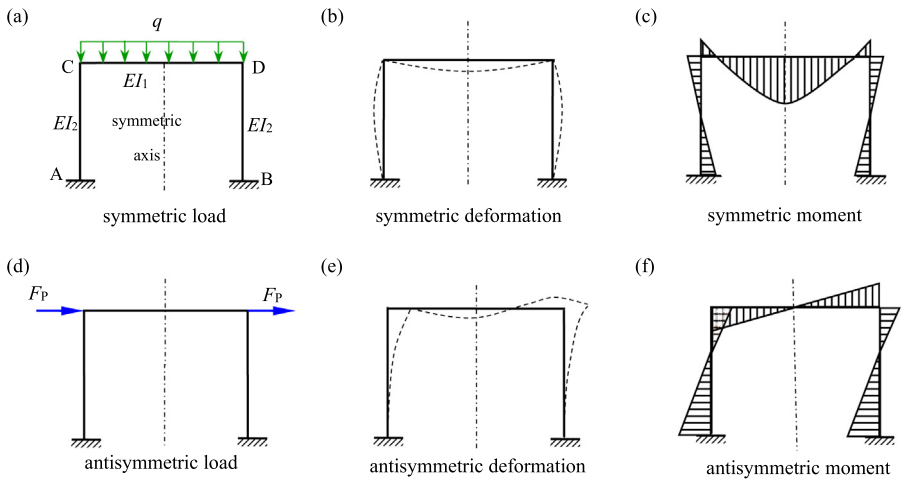


FIG. 5.23 – Deformation and moment diagrams of a symmetric frame under symmetric and antisymmetric loads.

By superimposing the loads in figure 5.24b and c, we can get the loads of the original structure. According to the principle of superposition, the deformation and internal forces of the original structure should be equal to the superposition of the results when the above two sets of loads are applied separately. It can be seen that the use of symmetry is possible as long as the structure is symmetric. Therefore, we can make use of the basic mechanical characteristics of the symmetric structure under symmetric and antisymmetric loads to simplify the calculation process.

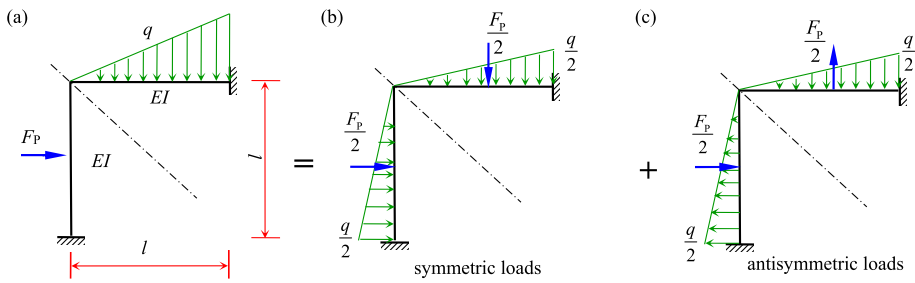


FIG. 5.24 – Decomposition of loads into symmetric loads and antisymmetric loads.

The symmetric or antisymmetric properties of the loads, displacements, and internal forces located on the symmetric axis of a symmetric structure can be determined as follows. Taking the symmetric frame shown in figure 5.25a for example, the vertical load F_{P1} on the symmetric axis is obviously a symmetric load, while the horizontal load F_{P2} and the moment M are antisymmetric loads. To understand this, consider the original load as two half loads that are infinitely close to

the axis of symmetry, and fold them about the axis of symmetry. Then the symmetry or antisymmetry of the load can be determined by whether the two half loads are in the same or inverse directions. In the same way, we can know that the vertical displacement on the symmetric axis section of the frame is a symmetric displacement, while the horizontal displacement and rotational angle are antisymmetric displacements. The internal forces of each cross-section of the structure are all in pairs, as shown in figure 5.25b. By checking the direction of the internal forces after folding about the axis of symmetry, it can be seen that the axial force on the axis of symmetry is symmetric, while the shear force is antisymmetric. The moment at the mid-span section of the horizontal beam is symmetric; while the moment in the middle vertical column is antisymmetric. The above conclusions are summarized in table 5.2.

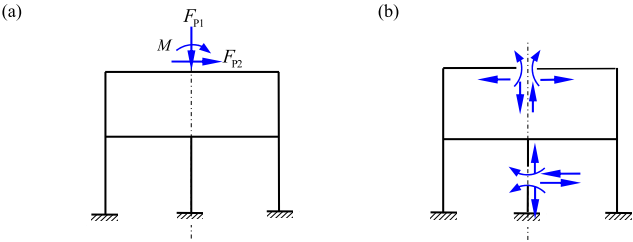


FIG. 5.25 – Loads, displacements and internal forces on the symmetric axis of a symmetric frame.

TAB. 5.2 – The properties of loads, displacements and internal forces on the axis of symmetry.

Type	Symmetry	Antisymmetry
Loads	Force along the symmetric axis	Force and moment perpendicular to the symmetric axis
Displacements	Linear displacement along the symmetric axis	Linear displacement and rotational angle perpendicular to the symmetric axis
Internal forces	Axial force, moment of horizontal members	Shear, moment of vertical members

It can be seen that under the action of symmetric loads, the shears of the members (including the horizontal and vertical members) on the symmetric axis and the moments of the members along the symmetric axis must be zero; under the action of antisymmetric loads, the axial forces of the members on the symmetric axis and the moments of the members perpendicular to the symmetric axis must be zero.

In the following section we will discuss how to use the above characteristics of symmetric structures to simplify the analysis and calculation of the force method.

5.5.1 Selection of Symmetric Primary System

The primary system of the force method is a system obtained after removing the redundant constraints from the original structure and substituting by the

corresponding unknown forces. Selecting a symmetric primary system can make some of the secondary coefficients and free terms in the force equations zero, thereby simplifying the calculation.

As shown in figure 5.26a is a symmetric portal frame. If the symmetric primary system shown in figure 5.26b is selected, the unit moment diagrams are shown in figure 5.26c, d, and e, respectively. The diagrams $\overline{M}_1$ and $\overline{M}_2$ under the symmetric unknown forces $X_1 = 1$ and $X_2 = 1$ are symmetric; the diagram $\overline{M}_3$ under the antisymmetric unknown force $X_3 = 1$ is antisymmetric. Therefore, we have

$$\delta_{13} = \delta_{31} = \sum \int \frac{\overline{M}_1 \overline{M}_3}{EI} ds = 0 \quad \delta_{23} = \delta_{32} = \sum \int \frac{\overline{M}_2 \overline{M}_3}{EI} ds = 0$$

Then, the force method equations can be simplified as

$$\left. \begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + A_{1P} &= 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + A_{2P} &= 0 \\ \delta_{33}X_3 + A_{3P} &= 0 \end{aligned} \right\} \quad (5.8)$$

It can be seen that if a symmetric primary system is selected for a symmetric structure, the force method equations are naturally decomposed into two groups. One group contains only symmetric unknown forces, and the other group contains only antisymmetric unknown forces. This is because symmetric unknown force does not cause antisymmetric displacement; likewise, antisymmetric unknown force does not cause symmetric displacement. This makes the associated secondary coefficients zero. Therefore, the force method equations are naturally decomposed into two uncoupled lower-order equations, simplifying the calculation. Under this condition, if the external load is symmetric, the antisymmetric unknown forces in the force method equations must be zero (as their corresponding free terms are zero); if the external load is antisymmetric, the symmetric unknown forces must be zero. In this way, the calculation can be further simplified. In the case of a general load, the load can be decomposed into symmetric loads and antisymmetric loads according to the needs of the solution, and then the problem can be solved by the principle of superposition.

The above fundamental principle of symmetry can be generalized and applied to the case that the primary unknowns are not on the axis of symmetry, by selecting a pair of unknown forces. For example, for the symmetric statically indeterminate frame shown in figure 5.27a, if we select the primary system in figure 5.27b, the corresponding unit moment diagram is not symmetric or antisymmetric. If the primary system in figure 5.27c is selected, the unknown vertical reactions at supports A and B can be regarded as the superposition of a set of symmetric unknown forces Y_1 and another set of antisymmetric unknown forces Y_2 . Then, the unit moment diagrams are shown in figure 5.27d and e, which are symmetric and antisymmetric, respectively. Therefore, the secondary coefficients of the force method equations $\delta_{12} = \delta_{21} = 0$, and the force method equations can be simplified into two uncoupled equations:

$$\begin{aligned} \delta_{11}Y_1 + A_{1P} &= 0 \\ \delta_{22}Y_2 + A_{2P} &= 0 \end{aligned}$$

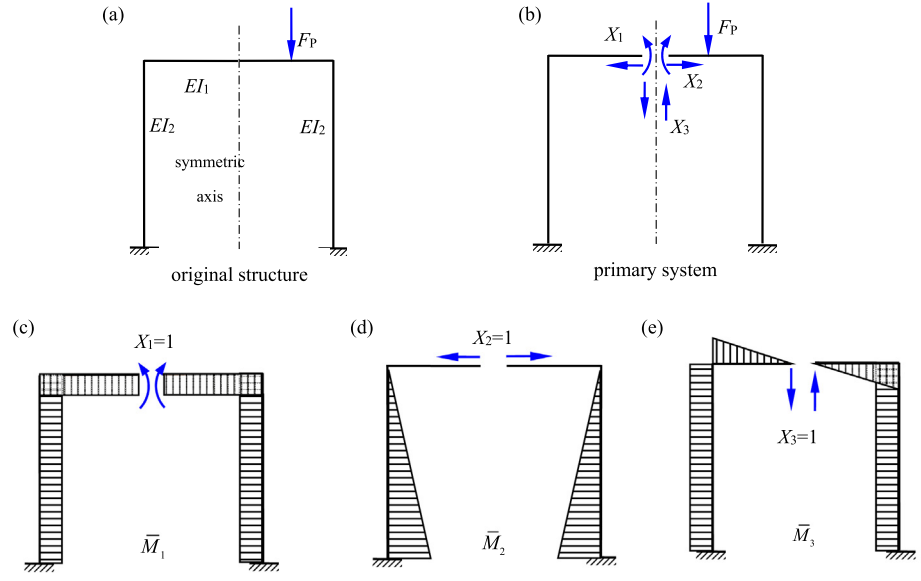


FIG. 5.26 – The unit moment diagrams of the symmetric primary system of a symmetric frame.

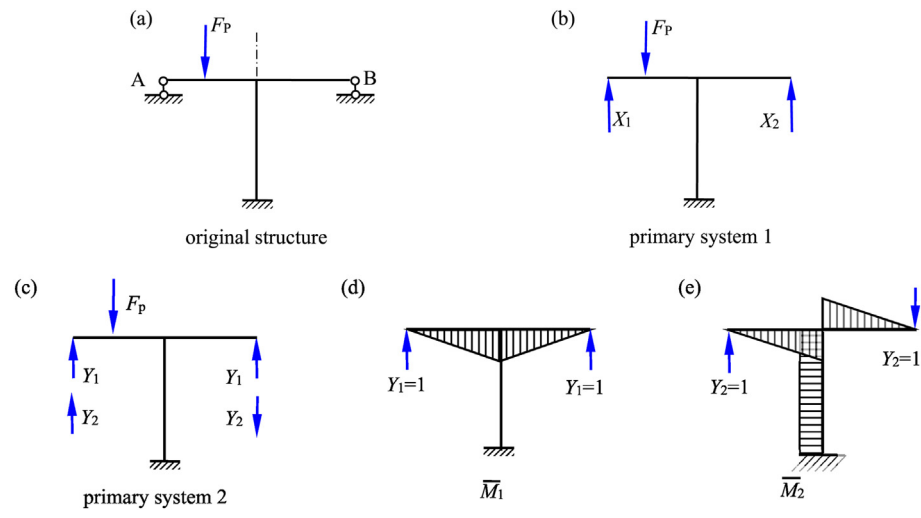


FIG. 5.27 – Selection of primary system by applying a pair of unknown forces.

The above Y_1 and Y_2 are two pairs of unknown generalized forces. The force method equations indicate that the generalized displacements corresponding to the generalized forces should be equal to zero. After solving Y_1 and Y_2 , the reactions X_1

and X_2 of supports A and B of the original structure can be obtained by the principle of superposition, that is,

$$X_1 = Y_1 + Y_2 \quad X_2 = Y_1 - Y_2$$

Example 5.5. Draw the moment diagram of the symmetric frame shown in figure 5.28a. EI is constant for each member.

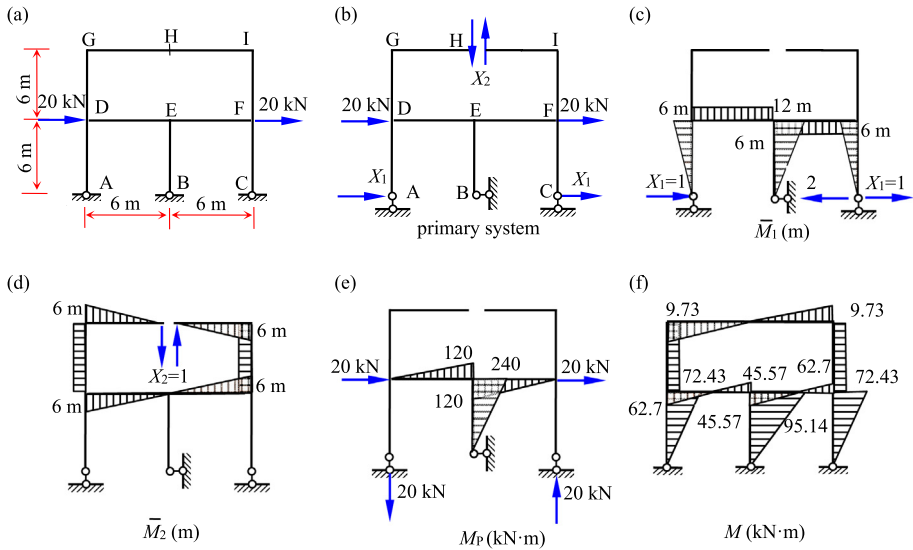


FIG. 5.28 – Analysis of a frame by using symmetric property.

Solution: This is a symmetric frame of 6 degrees of indeterminacy subjected to antisymmetric loads. In the analysis, we can remove the rigid constraint at section H on the symmetric axis, the horizontal constraints at supports A and C, and the vertical constraint at support B, and then the primary system is obtained as shown in figure 5.28b. The horizontal unknown constraint forces at supports A and C are set as paired unknown forces. As the external loads are antisymmetric, the symmetric constraint forces are zero, and only the antisymmetric constraint force X_1 exists. Support B is located on the axis of symmetry. The vertical constraint force is a symmetric force, so it must be zero. In the same way, there is only antisymmetric internal force X_2 at the notch of section H on the axis of symmetry. Therefore, by using symmetry, there are only two primary unknowns in the force method. The unit moment diagram and external load moment diagram are shown in figure 5.28c, d, and e, respectively. The force method equations are

$$\begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P} &= 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P} &= 0 \end{aligned}$$

The coefficients and free terms of the force method equations can be calculated by graph multiplication method, with the results as follows:

$$\delta_{11} = \frac{864}{EI} \text{ m}^3 \quad \delta_{22} = \frac{720}{EI} \text{ m}^3 \quad \delta_{12} = \delta_{21} = -\frac{216}{EI} \text{ m}^3$$

$$\Delta_{1P} = \frac{10080}{EI} \text{ kN} \cdot \text{m}^3 \quad \Delta_{2P} = -\frac{1449}{EI} \text{ kN} \cdot \text{m}^3$$

Substituting the above coefficients and free terms into the force method equations, we have

$$\begin{aligned} 864X_1 - 216X_2 + 10080 \text{ kN} &= 0 \\ -216X_1 + 720X_2 - 1449 \text{ kN} &= 0 \end{aligned}$$

Solving this equation set yields

$$X_1 = -12.07 \text{ kN}, \quad X_2 = -1.62 \text{ kN}$$

After obtaining the redundant constraint forces X_1 and X_2 , the moments of the members can be calculated according to the equilibrium conditions or the principle of superposition (5.7). Then, the moment diagram of the frame can be drawn, as shown in figure 5.28f.

In general, the main points of simplifying the calculation of structures by using the principle of symmetry can be summarized as follows:

- (1) Select the symmetric primary structure, and select the symmetric forces or antisymmetric forces as the primary unknowns;
- (2) Under symmetric loads, only symmetric unknown forces need to be considered (as the antisymmetric unknown forces and antisymmetric displacements are all equal to zero according to the principle of symmetry);
- (3) Under antisymmetric loads, only antisymmetric unknown forces need to be considered (the symmetric unknown forces and symmetric displacements are all equal to zero);
- (4) Asymmetric loads can be decomposed into symmetric loads and antisymmetric loads.

5.5.2 Determination of Half Structure in Terms of Symmetry

Utilizing the basic mechanical characteristics of a symmetric structure under symmetric and antisymmetric loads, the **half structure** can first be analyzed and calculated according to the principle that the deformation and internal forces should be equivalent to the original structure, and then the internal forces of the entire structure can be obtained through symmetry. In general, the degree of indeterminacy of the half structure is lower than that of the original structure, which can

simplify the calculation. The methods of selecting half structures are described below for odd-number-span and even-number-span symmetric structures.

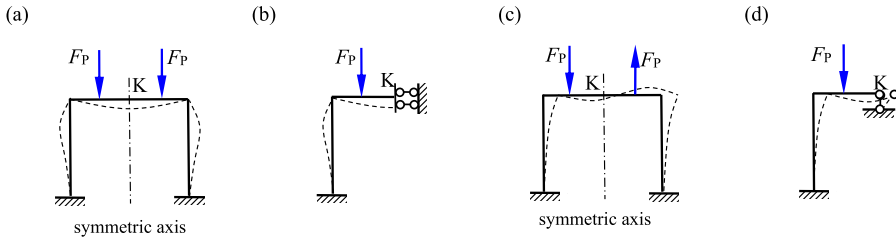


FIG. 5.29 – The half structure of a single-span symmetric frame.

Figure 5.29a shows a single-span symmetric frame subjected to symmetric loads. Under this condition, the deformation and internal forces of the frame should be symmetric. So there will be no horizontal displacement and angular displacement at section K on the axis of symmetry (*i.e.*, the antisymmetric displacements are zero), and only vertical displacement exists. Similarly, there can only exist moment and axial force at section K, with no shear (*i.e.*, the antisymmetric internal force is zero). Therefore, when calculating with the half structure, a slider support should be adopted at the symmetric section to replace the original rigid constraint. The computational model is shown in figure 5.29b. The deformation and internal forces should be the same as the original structure.

If the frame is subjected to antisymmetric loads, as shown in figure 5.29c, the deformation and internal forces should be antisymmetric. Under this condition, there will be no vertical displacement at section K on the axis of symmetry (*i.e.*, the symmetric displacement is zero), but horizontal displacement and angular displacement can exist. Similarly, there can only exist shear force at section K, without moment and axial force (*i.e.*, the symmetric internal forces are zero). Therefore, when calculating with the half structure, a vertical link (roller support) should be added at the symmetric section to replace the original connection. The computational model is shown in figure 5.29d.

For odd-number-span frames, the half structure can all be chosen according to the above principle to simplify the calculation. The case of even-number-span frames is discussed below.

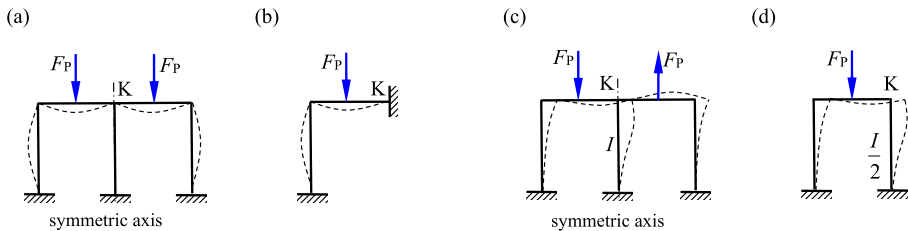


FIG. 5.30 – The half structure of a two-span symmetric frame.

Figure 5.30a shows a two-span symmetric frame subjected to symmetric loads. There should be no horizontal displacement and angular displacement at section K on the axis of symmetry, and there should be no vertical displacement after ignoring the axial deformation of the members. However, there are moments, shears, and axial forces on both sides of section K. When calculating with the half structure, a fixed support can be used to replace the connection. The computational model is shown in figure 5.30b. Under this condition, there will only exist axial force in the middle column of the frame, without moment and shear force (*i.e.*, the antisymmetric internal forces are zero). The value of the axial force should be double the vertical reaction of the fixed support at section K.

If the frame is subjected to antisymmetric loads, as shown in figure 5.30c, the half structure can be constructed by the following method: divide the middle column into two sub-columns that are infinitely close to each other, and the cross-sectional inertia moment of the sub-column should be half of the original column. Because the two sub-columns in the two half structures have only axial force, the computational model of the half structure is shown in figure 5.30d. When extending the internal forces of the half structure to the full structure according to the antisymmetric principle, it should be noted that the moment and shear of the middle column should be double the half structure. As the axial force of the middle column is a symmetric internal force, it must be zero after the superposition of the two half-structures.

Regardless of the odd-number-span or even-number-span symmetric frame, if the load is acting on the axis of symmetry, half of the load should be taken for the half structure, as the other half load will be carried by the other half structure.

Example 5.6. Draw the moment diagram of the symmetric frame shown in figure 5.31a. Each member has constant EI .

Solution: This frame has 4 degrees of indeterminacy, with two symmetric axes in the vertical and horizontal directions. The frame is even-number-span and odd-number-span in the two directions, respectively. According to the deformation and internal force characteristics of the frame under symmetric loads, a quarter of it can be taken for analysis. Noticing that the midpoints of the vertical members on both sides are hinged, the computational model should be as shown in figure 5.31b. Under this condition, the number of degrees of indeterminacy is reduced to one. The primary system can be taken as shown in figure 5.31c, and the moment diagram obtained by the force method is shown in figure 5.31d.

After obtaining the moment diagram of the quarter structure, the moment diagram of the full frame can be drawn according to the principle of symmetry, as shown in figure 5.31e. There is no moment and shear in the vertical member of the frame on the axis of symmetry, and the axial force is double the vertical reaction of the fixed support in the computational model.

Now we discuss the basic concepts related to example 5.6. The first is how to select the computational model if the influence of axial deformation of the members (the axial stiffness EA is constant) needs to be considered. Under this condition, the vertical bar on the axis of symmetry will undergo compression deformation. Obviously, the end of the vertical bar can no longer be regarded as a fixed support, but a

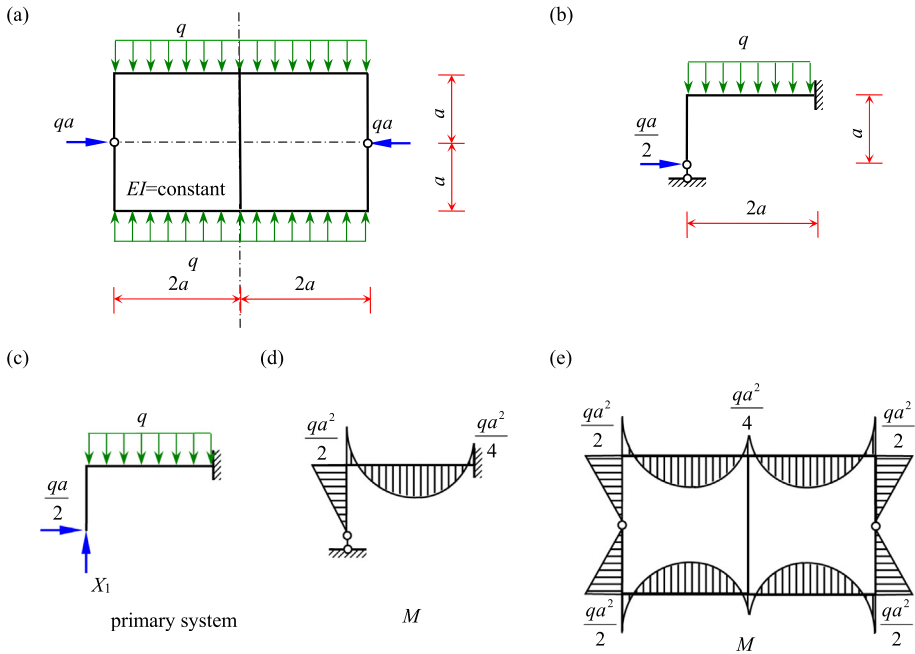


FIG. 5.31 – Analysis of a symmetric frame by using the half structure.

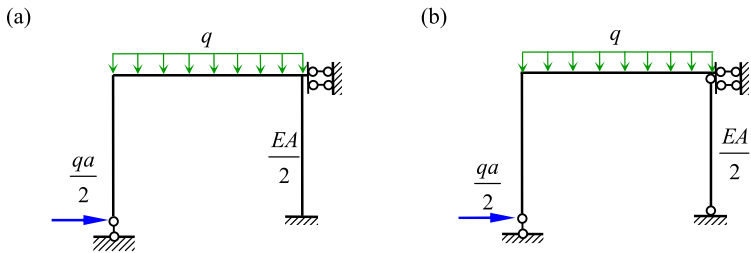


FIG. 5.32 – The half structure with a vertical bar.

slider support, and the vertical bar should be retained. When calculating with the half structure, the axial stiffness of the vertical bar should be taken as half of the original value. The computational model is shown in figure 5.32a or b.

Example 5.7. Draw the moment diagram of the symmetric frame shown in figure 5.33a. Each member has constant EI .

Solution: This frame is a symmetric structure with 4 degrees of indeterminacy. In order to make use of symmetry, the loads can be decomposed into symmetric and antisymmetric loads, as shown in figure 5.33b and c.

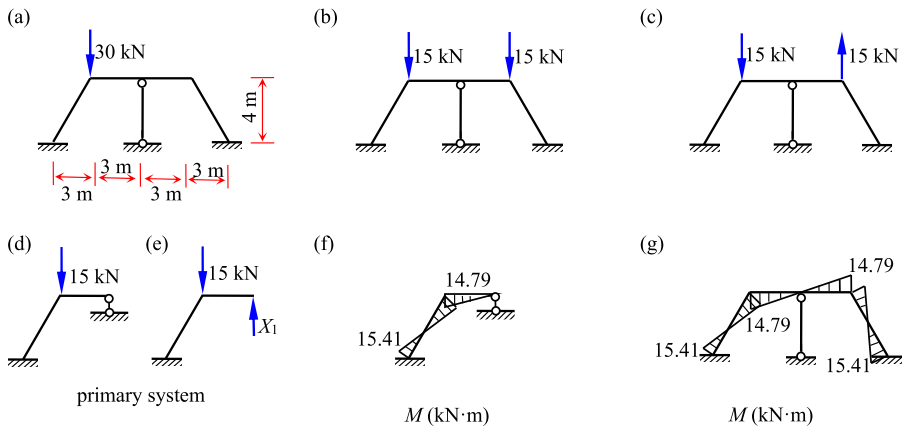


FIG. 5.33 – Analysis of a symmetric frame with 4 degrees of indeterminacy.

Under the action of the symmetric loads, there will be no linear and angular displacements at the joint after neglecting the axial deformation of the members. As the joint load is balanced by the axial force of the bar, there will be no moment and shear force in the frame. Under the action of antisymmetric loads, the axial force of the two-force member on the symmetric axis must be zero. The computational model of the half structure is shown in figure 5.33d, which has a single degree of indeterminacy. The primary system shown in figure 5.33e is chosen, and the moment diagram of the half structure can be solved by the force method, as shown in figure 5.33f. Then, the moment diagram of the original frame can be drawn according to the principle of antisymmetry of the internal force, as shown in figure 5.33g.

If the frame of example 5.7 includes a non-joint load in the symmetric loads, there will also exist moments in the frame under the symmetric loads. In this case, the moment diagram of the frame can be drawn by superimposing the moment diagrams of the symmetric and antisymmetric loads.

For statically indeterminate trusses, symmetry can also be used to simplify the calculations. Take the symmetric truss with a single degree of indeterminacy shown in figure 5.34a as an example. As the load is antisymmetric, it can be obtained that the internal force of member CD is zero, and the reactions at supports A and B are antisymmetric. Thus, it can be calculated according to the statically determinate truss shown in figure 5.34b.

If a structure is symmetric in geometry and material properties, but the support is asymmetric, then the reaction of the asymmetric support can be decomposed into symmetric and antisymmetric forces. When selecting the half structure, it should be noted that its deformation and internal forces should be identical to the original structure. For example, for the frame shown in figure 5.35a, the external loads and reaction of support D can be decomposed into a group of symmetric forces and a group of antisymmetric forces, as shown in figure 5.35b and c. Under the action of

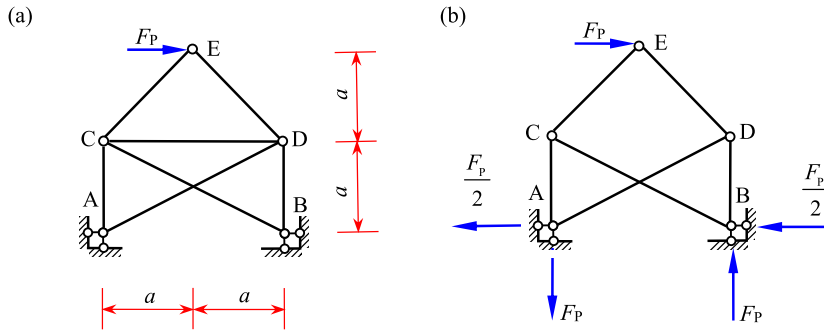


FIG. 5.34 – Calculation of a statically indeterminate truss by using symmetric property.

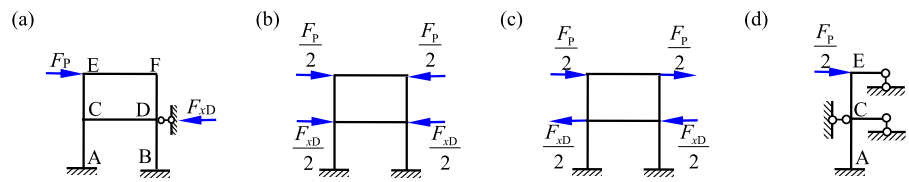


FIG. 5.35 – Decomposition of the asymmetric support reaction into symmetric and antisymmetric forces.

symmetric forces (figure 5.35b), there is only axial force in the members; under the action of antisymmetric forces (figure 5.35c), the half structure shown in figure 5.35d is taken for the calculation. Note that the horizontal support link should be added at point C under this condition. The internal forces of the frame can be obtained by the principle of superposition.

When using a computer to analyze a symmetric structure, the computational model of half structure can often be employed. In addition to the truss system, the principle of symmetry is also widely used in continuum mechanics and other types of structural numerical analysis. The key to the rational use of symmetry is to ensure that the deformation and mechanical characteristics of the model are completely consistent with the original structure.

5.6 Two-Hinged and Hingeless Arches

Arch structures are widely used in engineering. In the field of bridges, there is the famous Zhaozhou stone arch bridge with a long history. Hyperbolic arch bridges have been widely adopted in the past. In the field of construction, in addition to the floor-standing vault structures, arched roofs with tie bar are often used (in figure 5.36a, the arch is a reinforced concrete member, the tie bar is made of an angle steel, and the hang rods are used to prevent the tie bars from drooping.

The computational model is shown in figure 5.36b). Besides, the tunnel linings in water conservancy projects and underground buildings are also arch structures (figure 5.36c and d).

The statically indeterminate arches are mostly hingeless arches or two-hinged arches (figure 5.36e, f). A closed ring structure can be regarded as a special case of the hingeless arch.

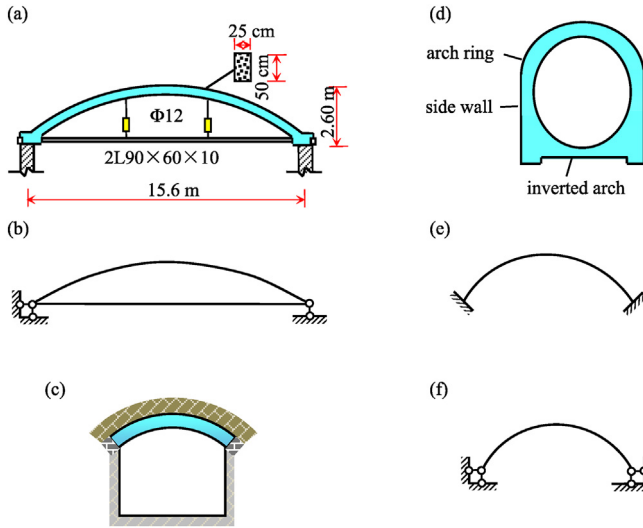


FIG. 5.36 – Various types of arch structures.

5.6.1 Solving Two-Hinged Arch by the Force Method

Two-hinged arch has a single degree of indeterminacy (figure 5.37a). The primary system can be selected as the corresponding simply supported curved beam, as shown in figure 5.37b. Taking the horizontal thrust X_1 at the support as the primary unknown, the force method equation can be established according to the deformation compatibility condition that the horizontal displacement of the original structure at support B should be zero:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

As the arch is curved, the displacements δ_{11} and Δ_{1P} should be calculated by the integral method, instead of the graph multiplication method.

Since the primary structure is a simply supported curved beam, generally only the influence of bending deformation needs to be considered in the calculation of Δ_{1P} . However, the axial deformation needs to be considered for flat arches with thick cross-section when calculating δ_{11} . Therefore,

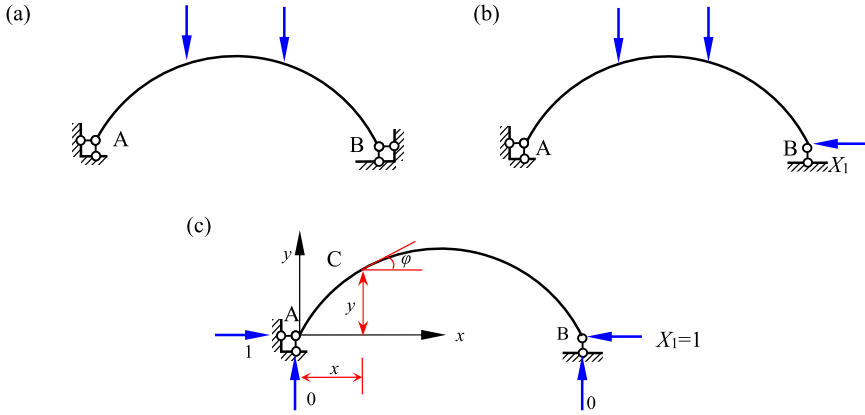


FIG. 5.37 – Solving the two-hinged arch by using force method.

$$\left. \begin{aligned} \Delta_{1P} &= \int \frac{\bar{M}_1 M_P}{EI} ds \\ \delta_{11} &= \int \frac{\bar{M}_1^2}{EI} ds + \int \frac{\bar{F}_{N1}^2}{EA} ds \end{aligned} \right\} \quad (a)$$

For the primary structure, under the action of $X_1 = 1$ (figure 5.37c), the vertical support reaction is zero, and the moment and axial force of an arbitrary section C on the arch axis are

$$\left. \begin{aligned} \bar{M}_1 &= -y \\ \bar{F}_{N1} &= -\cos \varphi \end{aligned} \right\} \quad (b)$$

where, y represents the ordinate of section C (upward is positive); φ represents the acute angle between the tangent of the arch axis at section C and the x -axis, which is positive for the left half arch and negative for the right half arch; the moment M is positive when the inner side of the arch is tensional; the axial force F_N is positive for tension.

If a two-hinged arch is only subjected to vertical loads, the moment M_P at any section of a simply supported curved beam is equal to the moment of the corresponding section of a simply supported horizontal beam with the same span and loads, *i.e.*

$$M_P = M^0 \quad (c)$$

Substituting equations (b) and (c) into (a), we get

$$\left. \begin{aligned} \Delta_{1P} &= - \int \frac{M^0 y}{EI} ds \\ \delta_{11} &= \int \frac{y^2}{EI} ds + \int \frac{\cos^2 \varphi}{EA} ds \end{aligned} \right\}$$

Once δ_{11} and Δ_{1P} are determined, X_1 (*i.e.*, the thrust F_H) can be solved by the force equation:

$$X_1 = F_H = -\frac{\Delta_{1P}}{\delta_{11}}$$

After the thrust F_H is obtained, the calculation of internal forces is exactly the same as the three-hinged arch. The formulas for the internal forces of two-hinged arches under vertical loads are

$$\left. \begin{aligned} M &= M^0 - F_H y \\ F_S &= F_S^0 \cos \varphi - F_H \sin \varphi \\ F_N &= -F_S^0 \sin \varphi - F_H \cos \varphi \end{aligned} \right\} \quad (5.9)$$

In the analysis and calculation process of the two-hinged arches, the following two points should be noted:

- (1) The thought of calculating a two-hinged arch using the force method is basically the same as a two-hinged frame. However, as the arch axis is a curve, the displacement influence coefficient δ_{11} and the free term Δ_{1P} of the two-hinged arch must be calculated by the integral method, instead of the graph multiplication method.
- (2) The mechanical characteristics of two-hinged arches are basically the same as those of three-hinged arches. The internal force calculation formula (5.9) is exactly the same as that of three-hinged arches, except for the value of F_H . For three-hinged arches, the thrust F_H is obtained from the equilibrium condition; for two-hinged arches, it is obtained from the deformation condition.

The two-hinged arches used in roof structures usually have a tie bar (figure 5.38a). The purpose of adding the tie bar, on one hand, is to avoid thrust in the brick wall or column, so that no moment is generated in the brick wall or column; on the other hand, is to make the arch rib bear thrust, thereby reducing the moment in the arch rib.

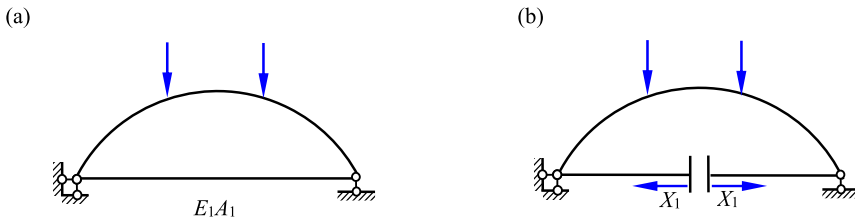


FIG. 5.38 – (a) Two-hinged arch with tie bar and (b) primary system.

To calculate a **two-hinged arch with a tie bar**, the tie bar can be cut off, with the primary system shown in figure 5.38b. The primary unknown of the force method is the tensile force X_1 in the tie bar, which is just the thrust F_H carried by the arch rib.

According to the deformation compatibility condition that the relative linear displacement on both sides of the notch of the tie bar should be zero, the force method equation can be established:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

Its form is the same as the two-hinged arch without a tie bar. However, when calculating δ_{11} , the influence of axial deformation of the tie bar should be taken into account, that is,

$$\delta_{11} = \int \frac{\overline{M}_1^2}{EI} ds + \int \frac{\overline{F}_{N1}^2}{EA} ds + \frac{\overline{F}_{N1}^2 l}{E_1 A_1} \quad (d)$$

where, the first two terms are the deformation terms of the arch rib, and the last term is the deformation term of the tie bar. E_1 and A_1 are the elastic modulus and cross-sectional area of the tie bar, respectively. l is the length of the tie bar. If $X_1 = 1$ acts on the primary structure, the axial force of the tie bar is $\overline{F}_{N1} = 1$. Therefore

$$\delta_{11} = \int \frac{\overline{M}_1^2}{EI} ds + \int \frac{\overline{F}_{N1}^2}{EA} ds + \frac{l}{E_1 A_1} \quad (e)$$

With loads acting on the primary structure, the axial force in the tie bar is zero. Therefore, the integral operation only needs to be performed in the arch rib for the calculation of Δ_{1P} , that is,

$$\Delta_{1P} = \int \frac{\overline{M}_1 M_P}{EI} ds \quad (f)$$

This formula is the same as the two-hinged arch without tie bar. The remaining solving steps are the same as above.

The analyses of two-hinged arches with and without tie bar are compared below. From equations (e) and (f), we can obtain the relationship between the displacements δ_{11} and Δ_{1P} of two-hinged arch without tie bar and the displacements δ_{11}^* and Δ_{1P}^* of two-hinged arch with tie bar:

$$\delta_{11}^* = \delta_{11} + \frac{l}{E_1 A_1}$$

$$\Delta_{1P}^* = \Delta_{1P}$$

Then the thrust F_H of two-hinged arch without tie bar and the thrust F_H^* of two-hinged arch with tie bar are expressed as follows:

$$F_H = -\frac{\Delta_{1P}}{\delta_{11}} \quad (g)$$

$$F_H^* = -\frac{\Delta_{1P}^*}{\delta_{11}^*} = -\frac{\Delta_{1P}}{\delta_{11} + \frac{l}{E_1 A_1}} \quad (h)$$

It can be seen from equation (h):

- (1) The thrust of two-hinged arch with a tie bar (*i.e.*, the tensile force of the tie bar) is usually smaller than the horizontal thrust of the support of the corresponding two-hinged arch without a tie bar.
- (2) When the tensile stiffness of the tie bar $E_1 A_1 \rightarrow \infty$, there is $F_H^* = F_H$, indicating the stress state of two-hinged arch with tie bar is the same as that of two-hinged arch without tie bar.
- (3) The thrust of the tie bar is directly related to the stiffness ratio of the tie bar and the arch rib. When the stiffness coefficient of the tie bar $\left(\frac{E_1 A_1}{l}\right)$ is much smaller than that of the arch rib $(k_{11} = \frac{1}{\delta_{11}})$, *i.e.*, $\frac{E_1 A_1}{l} \ll k_{11}$, then $F_H^* \rightarrow 0$ (for example, when $\frac{E_1 A_1}{l} \leq 0.1 k_{11}$, $F_H^* \leq 0.0909 F_H$). Under this condition, the two-hinged arch with a tie bar can be regarded as a simply-supported curved beam, and the static characteristics of the arch are lost, making the mechanical state of the arch rib very unfavorable. Therefore, when designing arches with tie bars, the axial stiffness of the tie bar should be appropriately increased to reduce the moment in the rib.

Example 5.8. Figure 5.39a shows a parabolic two-hinged arch subjected to an evenly distributed load in the half span. Determine the horizontal thrust F_H . The cross-sectional size of the arch is constant. The arch-axis equation with the left pivot point as the origin is

$$y = \frac{4f}{l^2} x(l - x)$$

Solution: The calculation can be simplified by using two assumptions:

- (1) Ignore the axial deformation of the arch, and only consider the influence of bending deformation.
- (2) When the arch is relatively flat (such as $\frac{f}{l} < \frac{1}{5}$), we can approximately take $ds = dx$ and $\cos \varphi = 1$. Then, the displacement formula is simplified as

$$\delta_{11} = \frac{1}{EI} \int_0^l y^2 dx$$

$$A_{1P} = -\frac{1}{EI} \int_0^l y M^0 dx$$

Calculate δ_{11} :

$$\delta_{11} = \frac{1}{EI} \int_0^l \left[\frac{4f}{l^2} x(l - x) \right]^2 dx = \frac{8f^2 l}{15EI}$$

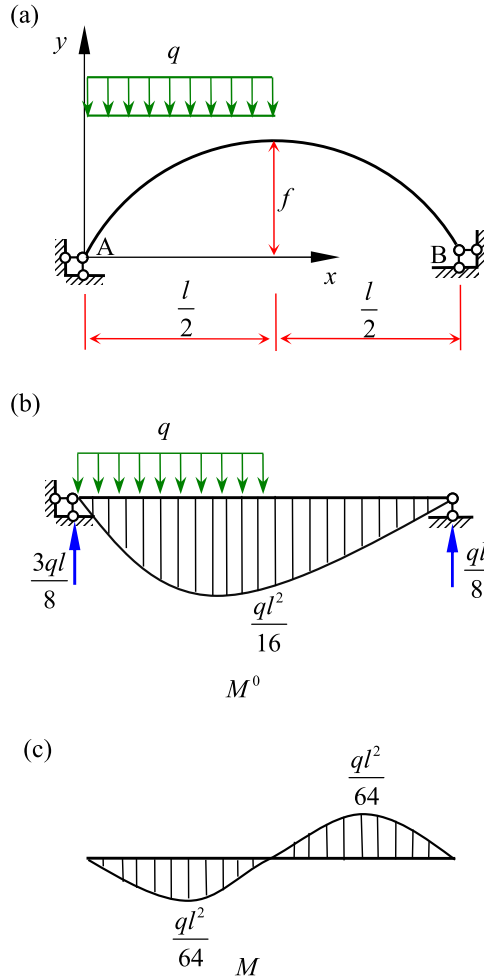


FIG. 5.39 – Solving a parabolic two-hinged arch with uniformly distributed load in half span.

To calculate Δ_{1P} , the moment M^0 of the corresponding simply supported beam needs to be calculated first. The diagram M^0 is shown in figure 5.39b, with the moment equation expressed in two segments as follows:

Left half span, $(0 < x < \frac{l}{2})$, $M^0 = \frac{3}{8}qlx - \frac{1}{2}qx^2$

Right half span, $(\frac{l}{2} < x < l)$, $M^0 = \frac{1}{8}ql(l - x)$

Therefore

$$\Delta_{1P} = -\frac{1}{EI} \int_0^{\frac{l}{2}} y \left(\frac{3}{8}qlx - \frac{1}{2}qx^2 \right) dx - \frac{1}{EI} \int_{\frac{l}{2}}^l y \frac{ql}{8} (l - x) dx = -\frac{ql^3 f}{30EI}$$

It can be obtained from the force method equation that

$$F_H = -\frac{\Delta_{1P}}{\delta_{11}} = \frac{ql^2}{16f}$$

This result is the same as the thrust of a three-hinged arch under a half-span uniform load.

After F_H is obtained, from the formula

$$M = M^0 - F_H y$$

The diagram M can be drawn, as shown in figure 5.39c. This moment diagram is also the same as the three-hinged arch.

However, it should be noted the conclusion that the thrust of a two-hinged arch is the same as the three-hinged arch (as obtained in the example) is not universal. If the loadings change or the influence of axial deformation cannot be ignored for calculating the displacement, the thrust of the two-hinged arch may be different from the three-hinged arch. Under general loads, the thrust of a two-hinged arch is usually close to the three-hinged arch.

5.6.2 Solving Hingeless Arch by the Force Method

Next, we will use the force method to solve common hingeless arch with symmetry. Figure 5.40a shows a symmetric **hingeless arch**, which has 3 degrees of indeterminacy. The following two measures can be applied to simplify the calculation.



FIG. 5.40 – (a) A symmetric hingeless arch and (b) symmetric primary system.

First, make use of the symmetry of the structure. Select a symmetric primary system by cutting off at the axis of symmetry of the vault, and take the bending moment X_1 , axial force X_2 , and shear force X_3 at the vault as the redundant unknown forces (figure 5.40b). X_1 and X_2 are symmetric unknown forces, and X_3 is an antisymmetric unknown force. Then the force method equation can be simplified into two independent groups:

$$\left. \begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1P} &= 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2P} &= 0 \\ \delta_{33}X_3 + \Delta_{3P} &= 0 \end{aligned} \right\} \quad (i)$$

Second, make use of the rigid arm. If the secondary coefficients in equation (i) can be made zero, *i.e.*, $\delta_{12} = \delta_{21} = 0$, the above force method equations can be simplified into three independent linear equations with only one unknown, then the internal forces can be easily solved. To achieve this, we can assume to move the action point of the unknown force on the vault section up and down in parallel. Under this condition, the unit moment diagram caused by $X_1 = 1$ and $X_3 = 1$ will not change, where the diagram $\overline{M}_1$ is the equivalent moment diagram of the arch rib with the inner side stretched. Originally, in the diagram $\overline{M}_2$ caused by $X_2 = 1$, the inner side of the whole arch is stretched, and the moment at the cut section is zero. When the action point of $X_2 = 1$ moves downward, the outer side of the arch rib in the middle becomes stretched, and the inner side of the arch rib at both ends remains stretched. Namely, the sign of the arch moment $\overline{M}_2$ changes, with both positive and negative values. Therefore, to make $\delta_{12} = 0$, the action position of $X_2 = 1$ should be moved downward.

Based on the above discussion, we can assume to cut the hingeless arch along the axis of symmetry at vault C, and attach two **rigid arms** of length y with infinite stiffness at both sides of the notch, then connect the two arms rigidly at the endpoint O, as shown in figure 5.41a. As the rigid arms will not deform, there will be no relative displacements (including relative linear displacement and relative angular displacement) between the left and right sections of cut section C under any load. In this way, the hingeless arch with rigid arms is completely equivalent to the original hingeless arch and can be replaced by each other.

Cut the hingeless arch with the rigid arms at the endpoint O of the arms, and take the two symmetric cantilever curved beams with a rigid arm as the primary structure (figure 5.41b). Take the moment X_1 , horizontal force X_2 , and vertical force X_3 at the cut section as the primary unknowns. According to the deformation compatibility condition that the relative displacement at point O should be zero, the force method equations can be established as equation (i).

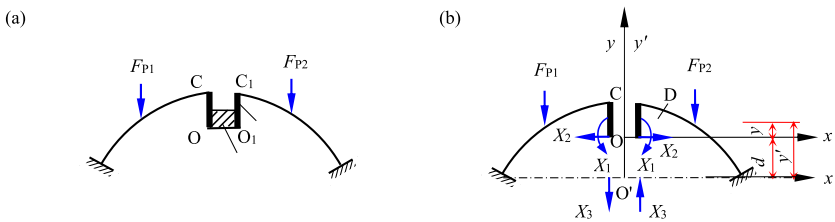


FIG. 5.41 – (a) The hingeless arch with rigid arms and (b) primary system.

Now, we need to determine the length of the rigid arm, *i.e.*, the position of endpoint O of the rigid arm. The purpose is to make the secondary coefficients of the force method equation $\delta_{12} = \delta_{21} = 0$. Then, the position of point O can be determined from this condition. To do this, let us first write the formula of δ_{12} as follows:

$$\delta_{12} = \int \frac{\overline{M}_1 \overline{M}_2}{EI} ds + \int \frac{k \overline{F}_{S1} \overline{F}_{S2}}{GA} ds + \int \frac{\overline{F}_{N1} \overline{F}_{N2}}{EA} ds \quad (j)$$

where, the internal forces under the unit load $X_1 = 1$ are

$$\left. \begin{aligned} \overline{M}_1 &= 1 \\ \overline{F}_{N1} &= 0 \\ \overline{F}_{S1} &= 0 \end{aligned} \right\} \quad (k)$$

The internal forces under the unit load $X_2 = 1$ are

$$\left. \begin{aligned} \overline{M}_2 &= -y \\ \overline{F}_{N2} &= -\cos \varphi \\ \overline{F}_{S2} &= -\sin \varphi \end{aligned} \right\} \quad (l)$$

Substituting equations (k) and (l) into equation (j) yields

$$\delta_{12} = - \int \frac{y}{EI} ds \quad (m)$$

In figure 5.41b, we can choose another reference coordinate axis $Ox'y'$: the y' axis coincides with the y -axis, and the distance between the x' -axis and x -axis is d . Then the new coordinate y' of any point D on the arch axis has the following relation with coordinate y :

$$y' = y + d \quad (n)$$

Substituting equation (n) into equation (m) yields

$$\delta_{12} = - \int \frac{y' - d}{EI} ds = - \int \frac{y'}{EI} ds + d \int \frac{1}{EI} ds$$

Let $\delta_{12} = 0$, we have

$$d = \frac{\int \frac{y'}{EI} ds}{\int \frac{1}{EI} ds} \quad (5.10)$$

When the equation of the arch axis and the changing rule of inertia moment of the arch rib cross-section are known, the length of the rigid arm, *i.e.*, the position of the endpoint O of the arm, can be determined from equation (5.10). The secondary coefficients among the three pairs of unknown forces at the left and right sections of cut O will all be zero. Point O is called the **elastic center** of the hingeless arch, whose position depends on the geometric parameters and physical characteristics of the structure. Figure 5.42a shows the actual hingeless arch. Figure 5.42b is an assumed band-shaped diagram along the arch axis curve with a width of $1/EI$, ds/EI represents the area of the differential element, and equation (5.10) is the computational formula of the centroid ordinate of the diagram. As the area of the diagram is related to the elastic property EI of the structure, it is called the elastic area. The endpoint of the rigid arm is just the centroid of the diagram, which is called the elastic center.

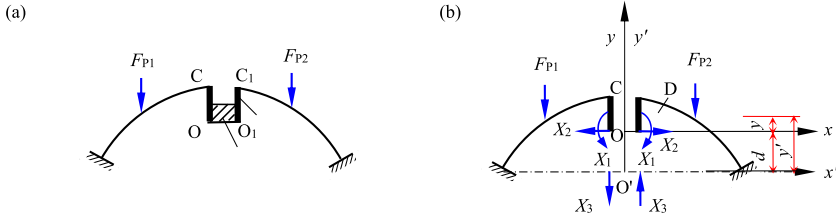


FIG. 5.42 – The elastic center of the hingeless arch.

Therefore, the solution of a symmetric hingeless arch can be summarized as follows: first, determine the position of the elastic center by equation (5.10); then select the two symmetric cantilever curved beams with a rigid arm as the primary structure, where the endpoint of the rigid arm is the elastic center; take the horizontal force, vertical force and moment at the endpoint of the arm as the primary unknowns; finally, solve the primary unknowns according to the force method equations. This method is called the **method of elastic center**.

In this case, the force method equations are simplified as:

$$\left. \begin{aligned} \delta_{11}X_1 + \Delta_{1P} &= 0 \\ \delta_{22}X_2 + \Delta_{2P} &= 0 \\ \delta_{33}X_3 + \Delta_{3P} &= 0 \end{aligned} \right\} \quad (5.11)$$

To calculate the displacements Δ_{iP} and δ_{ii} , usually only the influence of the moment needs to be considered; however, when calculating δ_{22} , sometimes the influence of the axial force also needs to be considered. Therefore, the following formulas are usually used for calculating the displacements:

$$\left. \begin{aligned} \Delta_{1P} &= \int \frac{\bar{M}_1 M_P}{EI} ds, \quad \delta_{11} = \int \frac{\bar{M}_1^2}{EI} ds \\ \Delta_{2P} &= \int \frac{\bar{M}_2 M_P}{EI} ds, \quad \delta_{22} = \int \frac{\bar{M}_2^2}{EI} ds + \int \frac{\bar{F}_{N2}^2}{EA} ds \\ \Delta_{3P} &= \int \frac{\bar{M}_3 M_P}{EI} ds, \quad \delta_{33} = \int \frac{\bar{M}_3^2}{EI} ds \end{aligned} \right\} \quad (5.12)$$

Substituting the expressions of $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ and $\bar{F}_{N2}$ into equation (5.12), we get

$$\left. \begin{aligned} \Delta_{1P} &= \int \frac{M_P}{EI} ds, \quad \delta_{11} = \int \frac{1}{EI} ds \\ \Delta_{2P} &= - \int \frac{y M_P}{EI} ds, \quad \delta_{22} = \int \frac{y^2}{EI} ds + \int \frac{\cos^2 \varphi}{EA} ds \\ \Delta_{3P} &= \int \frac{x M_P}{EI} ds, \quad \delta_{33} = \int \frac{x^2}{EI} ds \end{aligned} \right\} \quad (5.13)$$

It should be noted that, as many statically indeterminate arches (including two-hinged arches and hingeless arches) have variable cross-section, and the arch

axis is a curve, generally the numerical integration is used to calculate the primary coefficient δ_{ij} and the free term Δ_{iP} , by the segmental superposition method.

5.7 Internal Force Analysis of Structures under Support Movement or Temperature Change

In addition to bearing loads, engineering structures are often affected by factors such as support displacement, temperature change, material shrinkage, or manufacturing error of structural members, which can be collectively referred to as non-load factors. Whether a structure generates internal forces under non-load factors depends entirely on whether it is fully free to deform when subjected to these factors. If it can deform freely, no internal force will be caused, otherwise, the structure will generate internal forces. In statically determinate structures, although these non-load factors will cause displacement and deformation, they will not generate internal forces. In statically indeterminate structures, these factors will all generate internal forces. This is also an important difference between statically indeterminate and statically determinate structures. The internal forces generated in a statically indeterminate structure under the action of support movement or temperature change are called the **self-internal forces**.

The fundamental principle and steps of analyzing a statically indeterminate structure under non-load factors by the force method are the same as that under loads. The difference is that the free term in the force method canonical equation is no longer caused by the loads, but is the displacement of the primary structure in the direction of redundant constraint force caused by support movement, temperature change, or manufacturing error. Large internal forces may generate in engineering structures under these non-load factors, and sometimes can even cause structural damage. In engineering design, it is usually necessary to take measures according to the requirements of relevant design specification to reduce the influence of non-load factors, such as setting expansion joints for super-long structures, employing settlement joints between different structures and sections, etc., with the effects of non-load factors fully considered.

The detailed calculation process of analyzing the self-internal forces of a statically indeterminate structure under support movement or temperature change is described below, and the differences from that under loads are discussed.

5.7.1 Support Movement

To calculate the internal forces of a structure with n degrees of indeterminacy under the action of support movement, the general form of the i th equation in the force method canonical equations can be written as

$$\sum_{j=1}^n \delta_{ij} X_j + \Delta_{ic} = \Delta_i \quad (5.14)$$

where, δ_{ij} is the flexibility coefficient; Δ_{ic} indicates the displacement of the primary structure in the direction of X_i under the action of support displacement; and Δ_i

indicates the displacement of the original structure in the direction of X_i . All the above parameters are positive when the direction is in accordance with the assumed unknown force, otherwise negative. Again, the physical meaning of the above force method equation is: the displacement of the primary structure in the direction of the redundant constraint force should be consistent with the original structure under the combined action of each redundant constraint force and the support displacement.

Example 5.9. Figure 5.43a shows a beam AB with uniform cross-section. The left end is a fixed support and the right end is a roller support. If the rotation angle θ at the left end and the settlement displacement a of the right end support is known, determine the self-internal forces in the beam.

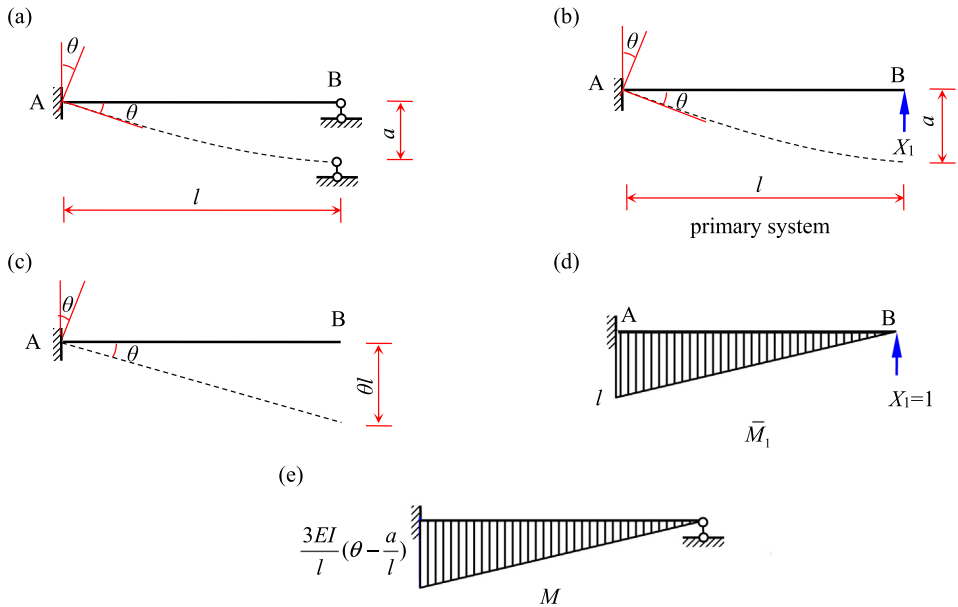


FIG. 5.43 – Solving the self-internal forces in a beam under support movement.

Solution: This beam has a single degree of indeterminacy. Take the vertical reaction of support B as the redundant unknown force X_1 , and then the primary system is a cantilever beam (figure 5.43b).

The deformation compatibility condition is: the vertical displacement Δ_1 of the primary system at point B should be the same as the original structure. Since the vertical displacement of the original structure at point B is known as a and the direction is opposite to X_1 , the deformation compatibility condition becomes:

$$\Delta_1 = -a \quad (a)$$

Considering that the displacement Δ_1 of the primary system is caused by the combined action of the unknown force X_1 and the rotation angle θ of support A, equation (a) can be written as

$$\delta_{11}X_1 + \Delta_{1c} = -a \quad (b)$$

where, the physical meaning of the free term Δ_{1c} on the left side of the equation is the displacement generated in the primary structure along the X_1 direction when support A has a rotation angle θ . From figure 5.43c, we have

$$\Delta_{1c} = -\theta l \quad (c)$$

The coefficient δ_{11} can be obtained from the diagram $\overline{M}_1$ in figure 5.43d

$$\delta_{11} = \frac{1}{EI} \int \overline{M}_1^2 dx = \frac{l^3}{3EI} \quad (d)$$

Substituting equations (c) and (d) into equation (b) yields

$$\frac{l^3}{3EI} X_1 - \theta l = -a \quad (e)$$

Then we have

$$X_1 = \frac{3EI}{l^2} \left(\theta - \frac{a}{l} \right) \quad (f)$$

As the primary structure is a statically determinate structure, the support movement does not cause internal forces in the primary structure, and the internal forces are only caused by the redundant unknown force. Then the superposition formula of the moment becomes

$$M = \overline{M}_1 X_1 \quad (g)$$

The diagram M is shown in figure 5.43e.

There are two points that need to be stated:

(1) Calculation characteristics under support movement

Compared with the case under loads, it can be seen from equation (a) or (b) that the right term of the force method equation may be non-zero. From equations (b) and (c), we can see that the free term Δ_{1c} of the force method equation is generated by the support movement. From equation (g), we can see that the internal forces are all caused by the redundant unknown force. It can be further seen from equations (f) and (g) that the internal forces are related to the absolute value of the stiffness EI of the members.

(2) Calculation with different primary structures

If the simply supported beam is taken as the primary system and the reaction moment at support A is taken as the redundant unknown force X_1 (figure 5.44a), the deformation compatibility condition is: the rotation angle of the simply supported

beam at point A should be equal to the given value θ . Then the force method equation becomes

$$\delta_{11}X_1 + \Delta_{1c} = \theta$$

The free term Δ_{1c} is the rotation angle of the simply supported beam at point A caused by the settlement displacement a of support B. From figure 5.44b we have

$$\Delta_{1c} = \frac{a}{l}$$

The coefficient δ_{11} can be obtained from the diagram $\overline{M}_1$ in figure 5.44c

$$\delta_{11} = \frac{l}{3EI}$$

Then the force method equation becomes

$$\frac{l}{3EI}X_1 + \frac{a}{l} = \theta \quad (\text{h})$$

Thus

$$X_1 = \frac{3EI}{l} \left(\theta - \frac{a}{l} \right)$$

Similarly, the diagram M of the structure can be solved, as shown in figure 5.43e.

For the same statically indeterminate structure, two different primary structures can be selected, and then two different force equations (e) and (h) are obtained. Each force method equation contains the two support displacement parameters θ and a .

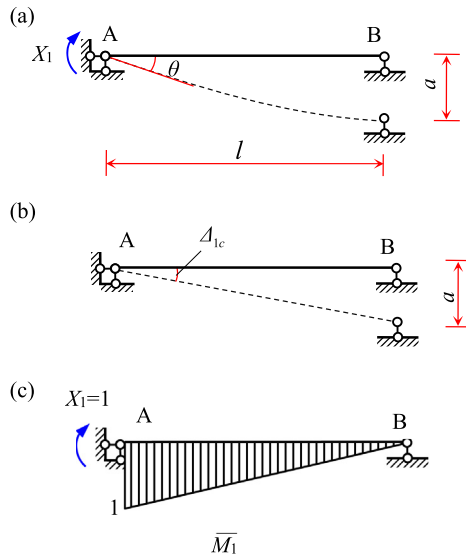


FIG. 5.44 – Calculation with simply supported beam as the primary structure.

In equation (e), θ is on the left and a is on the right; but in equation (h), θ is on the right and a is on the left. Generally speaking, all the support displacement parameters corresponding to the redundant unknown force appear in the right terms of the force method equation, and other support displacement parameters appear in the free terms of left side.

If the primary system is selected as shown in figure 5.45, both the support displacement parameters θ and a are not related to X_1 . Therefore, they both appear in the left free terms of the force method equation, and the right term of the force method equation is zero.

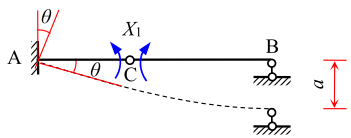


FIG. 5.45 – Another selection of primary system.

5.7.2 Temperature Change

Example 5.10. Figure 5.46a shows a reinforcement concrete frame. The pouring temperature is 15 °C. In winter the outer surface temperature of the concrete is

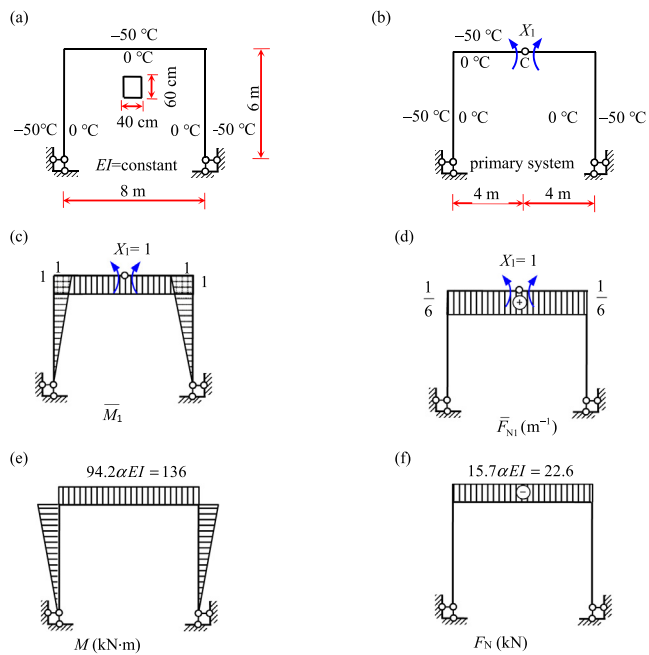


FIG. 5.46 – Analysis of a reinforcement concrete frame under temperature change.

-35°C , and the inner surface temperature is 15°C . Determine the internal forces in the frame caused by the temperature change. The EI of each member is constant, and the cross-sectional dimensions are shown in the figure. The elastic modulus of the concrete is $E = 2 \times 10^{10} \text{ MPa}$; the coefficient of linear expansion is $\alpha = 1 \times 10^{-5} \text{ }^{\circ}\text{C}^{-1}$.

Solution: (1) Primary system

This frame has a single degree of indeterminacy. The primary system can be selected as shown in figure 5.46b.

(2) Force method equation

The deformation compatibility condition is: the relative rotation angle of the primary system at hinge C should be zero. This displacement is caused by the temperature change and the unknown force X_1 together, that is,

$$\delta_{11}X_1 + \Delta_{1t} = 0$$

where, the free term Δ_{1t} is the displacement of the primary structure along the X_1 direction caused by the temperature change.

(3) Coefficients and free terms

The calculation method of the coefficient δ_{11} is the same as that under loads, but the free term Δ_{1t} is different.

According to the pouring temperature and winter temperature, the temperature change at the outer side of the frame is: $-35^{\circ}\text{C} - 15^{\circ}\text{C} = -50^{\circ}\text{C}$, and the temperature change at the inner side is: $15^{\circ}\text{C} - 15^{\circ}\text{C} = 0^{\circ}\text{C}$.

The average temperature change on the axis is $t_0 = 0.5 \times (0^{\circ}\text{C} - 50^{\circ}\text{C}) = -25^{\circ}\text{C}$, and the temperature difference between the inner and outer sides is $0^{\circ}\text{C} - (-50^{\circ}\text{C}) = 50^{\circ}\text{C}$.

The calculation formula of Δ_{1t} is

$$\Delta_{1t} = \sum \alpha \frac{\Delta t}{h} \int \bar{M}_1 ds + \sum \alpha t_0 \int \bar{F}_{N1} ds \quad (\text{i})$$

where, the two integrals are the area of the diagram $\bar{M}_1$ and diagram $\bar{F}_{N1}$, respectively.

Draw the diagram $\bar{M}_1$ and diagram $\bar{F}_{N1}$ (figure 5.46c and d). According to the displacement formula we have

$$\delta_{11} = \sum \int \frac{\bar{M}_1^2}{EI} ds = \frac{1}{EI} \times \left[(1 \times 8) \times 1 + 2 \times \left(\frac{1}{2} \times 1 \times 6 \right) \times \frac{2}{3} \right] = \frac{12}{EI}$$

$$\Delta_{1t} = \alpha \times \frac{50}{0.6} \times \left(1 \times 8 + \frac{1 \times 6}{2} \times 2 \right) - \alpha \times 25 \times \left(\frac{1}{6} \times 8 \right) = 1166\alpha - 33\alpha = 1133\alpha$$

In the diagram $\bar{M}_1$, the inner side of the frame is in tension, and the temperature difference Δt has higher temperature at the inner side. Therefore, the first term in

the above equation is positive. In the diagram $\overline{F}_{N1}$, the horizontal beam is in tension and the temperature change t_0 is negative, so the second term in the above equation is negative.

(4) Solve the force method equation

From the force method equation, we have

$$X_1 = -\frac{\Delta_{1t}}{\delta_{11}} = -\frac{1133\alpha}{12/EI} = -92.4\alpha EI$$

(5) Draw the internal force diagrams

As the primary structure is statically determinate, temperature change will cause no internal force. Thus, internal forces are all caused by the redundant unknown force, that is,

$$\left. \begin{aligned} M &= \overline{M}_1 X_1 \\ F_N &= \overline{F}_{N1} X_1 \end{aligned} \right\}$$

The internal force diagrams are shown in figure 5.46e and f.

It can be seen from the calculation results that the cross-sectional stiffness of the members is included in the internal forces and reactions of the statically indeterminate structure under temperature change, indicating that the internal and reaction forces are proportional to the absolute value of the stiffness of the members. This is different from the case under loads, where the internal and reaction forces of a statically indeterminate structure are only related to the relative stiffness of the members. Therefore, under a given temperature change, the larger the cross-sectional size, the greater the internal forces. This means increasing the cross-sectional size is not an effective way for improving the stress condition of a structure under temperature change. In addition, when there is a temperature difference Δt in the structure members, the positive value of the moment diagram appears on the cooling side, which means compressive stress appears on the heating side and tensile stress on the cooling side. Therefore, in reinforced concrete structures, special attention should be paid to possible cracks under cooling.

5.8 Deflection Computation of Statically Indeterminate Structures

The unit load method, as an application of the principle of virtual work for deformable structures, not only can be used to compute the deflection of statically determinate structures but also the **deflection of statically indeterminate structure**.

Taking the statically indeterminate beam shown in figure 5.47a as an example, the deflection f of the midpoint C under the uniform load can be calculated by the unit load method.

First of all, when analyzing a statically indeterminate structure by the force method, an arbitrary statically determinate primary system needs to be established. The primary system bears the combined action of external loads and redundant constraint forces, as shown in figure 5.47b. Then, establish the force method equation, solve the internal forces, and draw the moment diagram M (figure 5.47c). The stress condition and deformation of the primary system are completely consistent with the original statically indeterminate structure. Therefore, the displacement of the primary system is the same as the original statically indeterminate structure. Finally, the displacement of the primary system can be calculated by the unit load method. In addition to the load moment diagram of the statically indeterminate structure, a unit moment diagram of the virtual state also needs to be drawn. The virtual unit load is applied to the primary structure and is related to the displacement to be determined. This unit load can also be applied to the original statically indeterminate structure, but it is much more tedious to draw the unit moment diagram than it is for the primary structure. Therefore, this method is not a good choice to calculate the displacement.

In the following example, the vertical displacement of point C is to be determined. We can apply a unit vertical load at point C on the primary structure (simple supported beam), and then the unit moment diagram $\bar{M}$ is easy to draw, as shown in figure 5.47d. By multiplying the two moment diagrams $\bar{M}$ and M , the displacement of a given section of the structure can be obtained.

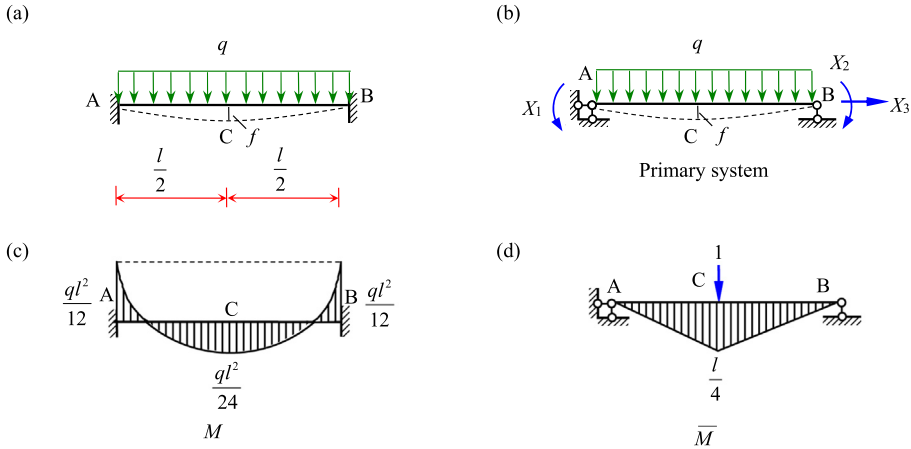


FIG. 5.47 – Calculation of deflection of a statically indeterminate beam.

The deflection f of point C of the original structure obtained by using the primary structure is:

$$\begin{aligned}
 f &= \int \frac{\bar{M}M}{EI} ds = \frac{2}{EI} \times \left[-\left(\frac{ql^2}{12} \times \frac{l}{2}\right) \times \left(\frac{1}{2} \times \frac{l}{4}\right) + \left(\frac{2}{3} \times \frac{ql^2}{8} \times \frac{l}{2}\right) \times \left(\frac{5}{8} \times \frac{l}{4}\right) \right] \\
 &= \frac{ql^4}{384EI} (\downarrow)
 \end{aligned}$$

It is worth noting that different primary structures can be used for solving a statically indeterminate structure. Therefore, there are various types of unit internal force diagrams that can be used to calculate the displacement of a statically indeterminate structure. For example, when computing the mid-span displacement f of a beam fixed at both ends, the unit moment diagram shown in figure 5.48a or b can also be used. Although the unit moment diagram is different, the obtained displacement should be the same.

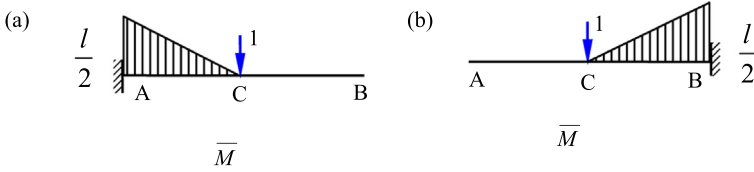


FIG. 5.48 – Selection of different unit moment diagrams.

The general formula for calculating the displacement of a plane member structure is:

$$\Delta = \sum \int (\bar{M}\kappa + \bar{F}_N\varepsilon + \bar{F}_S\gamma_0)ds - \sum \bar{F}_{RK}c_K$$

This formula is applicable to both statically determinate and indeterminate structures. The calculation formulas of the displacements of statically indeterminate structures under the action of external loads, support movement and temperature change are given below.

(1) External loads

Assuming the internal forces of a statically indeterminate structure under external loads is M , F_N and F_S , the deformation of the differential element of the structure member is

$$\kappa = \frac{M}{EI}, \quad \varepsilon = \frac{F_N}{EA}, \quad \gamma_0 = \frac{kF_S}{GA}$$

Thus, the displacement formula is

$$\Delta = \sum \int \frac{\bar{M}M}{EI}ds + \sum \int \frac{\bar{F}_N F_N}{EA}ds + \sum \int \frac{k\bar{F}_S F_S}{GA}ds \quad (5.15)$$

The form of equation (5.15) is exactly the same as the displacement formula of statically determinate structures. However, the $\bar{M}$, $\bar{F}_N$ and $\bar{F}_S$ here can be the internal forces of any primary structure under the unit load (it can also be the primary structure selected for calculating the internal forces of the statically indeterminate structure).

(2) Support movement

Assuming the internal forces of the statically indeterminate structure under support movement is M , F_N and F_S , the deformation of the differential element of the structure member is still

$$\kappa = \frac{M}{EI}, \varepsilon = \frac{F_N}{EA}, \gamma_0 = \frac{kF_S}{GA}$$

Thus, the displacement formula is

$$\Delta = \sum \int \frac{\overline{M}M}{EI} ds + \sum \int \frac{\overline{F}_N F_N}{EA} ds + \sum \int \frac{k\overline{F}_S F_S}{GA} ds - \sum \overline{F}_{RK} c_K \quad (5.16)$$

(3) Temperature change

Assume the internal forces of the statically indeterminate structure under temperature change is M , F_N and F_S . Under this condition, in addition to the elastic deformation caused by the internal forces, there is also deformation caused by the free expansion of the differential element under the temperature change, that is

$$\kappa = \frac{M}{EI} + \frac{\alpha \Delta t}{h}, \varepsilon = \frac{F_N}{EA} + \alpha t_0, \gamma_0 = \frac{kF_S}{GA}$$

Thus, the displacement formula becomes

$$\begin{aligned} \Delta = & \sum \int \frac{\overline{M}M}{EI} ds + \sum \int \frac{\overline{F}_N F_N}{EA} ds + \sum \int \frac{k\overline{F}_S F_S}{GA} ds \\ & + \sum \int \frac{\overline{M}\alpha \Delta t}{h} ds + \sum \int \overline{F}_N \alpha t_0 ds \end{aligned} \quad (5.17)$$

(4) Displacement formula under multiple factors

If the statically indeterminate structure is under the combined action of loads, support movement and temperature change, etc., the general formula for displacement calculation can be written as

$$\begin{aligned} \Delta = & \sum \int \frac{\overline{M}M}{EI} ds + \sum \int \frac{\overline{F}_N F_N}{EA} ds + \sum \int \frac{k\overline{F}_S F_S}{GA} ds \\ & + \sum \int \frac{\overline{M}\alpha \Delta t}{h} ds + \sum \int \overline{F}_N \alpha t_0 ds - \sum \overline{F}_{RK} c_K \end{aligned} \quad (5.18)$$

where, M , F_N and F_S represent the internal forces of statically indeterminate structure under the action of all factors, while $\overline{M}$, $\overline{F}_N$, $\overline{F}_S$ and $\overline{F}_{RK}$ are the internal forces and support reaction of primary structure under unit load.

Example 5.11. Determine the horizontal displacement of point C of the statically indeterminate frame shown in figure 5.49a.

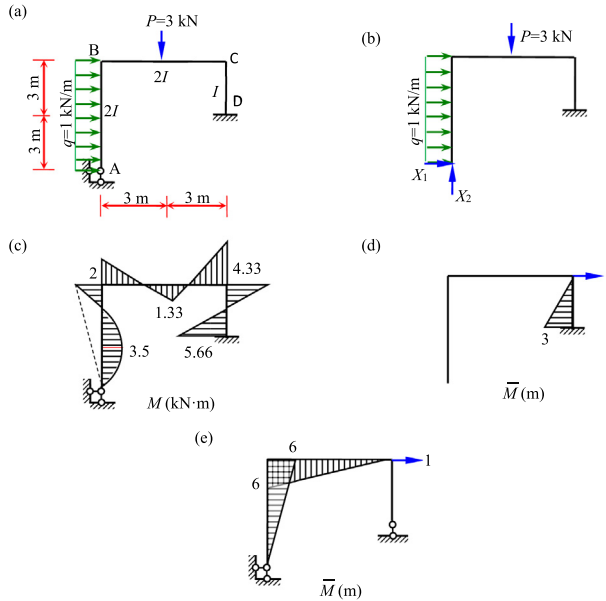


FIG. 5.49 – Solving the displacement of a statically indeterminate frame.

Solution: The primary system is shown in figure 5.49b, and the primary unknowns of the force method are X_1 and X_2 . It can be solved that $X_1 = -2.67$ kN and $X_2 = 1.11$ kN. The load moment diagram M of the statically indeterminate frame is displayed in figure 5.49c.

To calculate the horizontal displacement of point C, apply a horizontal unit load at point C of the primary structure, and the unit moment diagram of the primary structure is shown in figure 5.49d. Through graphical multiplying of figure 5.49c and d, the horizontal displacement of point C is obtained by the following formula

$$\Delta_C^H = \sum \int \frac{M\bar{M}}{EI} dx$$

$$\begin{aligned} \Delta_C^H &= \frac{1}{EI} \frac{1}{2} \times 3 \text{ m} \times 3 \text{ m} \times \left(\frac{2}{3} \times 5.66 \text{ kN}\cdot\text{m} - \frac{1}{3} \times 4.33 \text{ kN}\cdot\text{m} \right) \\ &= \frac{10.485 \text{ kN}\cdot\text{m}^3}{EI} (\rightarrow) \end{aligned}$$

If the primary system shown in figure 5.49e is selected, after drawing the corresponding unit moment diagram and graphically multiplying the moment diagrams of the original structure, the horizontal displacement of point C can also be obtained. The result is consistent with that of the previous selected primary system.

Example 5.12. Determine the mid-span deflection of the statically indeterminate beam in example 5.9 under the action of support displacement.

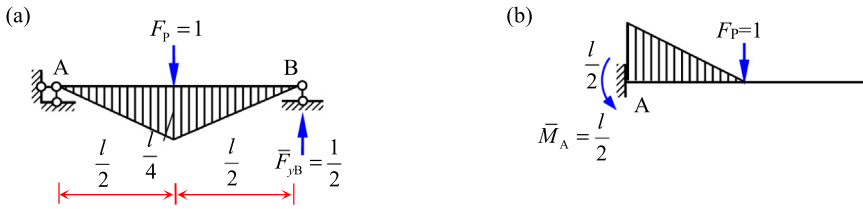


FIG. 5.50 – Solving the mid-span deflection of a statically indeterminate beam.

Solution: The moment diagram M of the statically indeterminate beam caused by the support displacement is shown in figure 5.43e.

To draw the unit moment diagram, the following two primary structures can be selected:

- (1) Take the simply supported beam as the primary structure. Figure 5.50a shows the diagram $\bar{M}$ and the reaction under the unit load.

$$\begin{aligned} \Delta &= \int \frac{\bar{M}M}{EI} ds - F_{yB}(-a) \\ &= \frac{1}{EI} \times \left(\frac{1}{2} \times \frac{l}{4} \times l \right) \times \left[\frac{1}{2} \times \frac{3EI}{l} \left(\theta - \frac{a}{l} \right) \right] - \frac{1}{2}(-a) = \frac{3}{16} \theta l + \frac{5}{16} a \end{aligned}$$

- (2) Take the cantilever beam as the primary structure. Figure 5.50b indicates the diagram $\bar{M}$ and the reaction under the unit load.

$$\begin{aligned} \Delta &= \int \frac{\bar{M}M}{EI} ds - \bar{M}_A(-\theta) \\ &= \frac{-1}{EI} \times \left(\frac{1}{2} \times \frac{l}{2} \times \frac{l}{2} \right) \times \left[\frac{5}{6} \times \frac{3EI}{l} \left(\theta - \frac{a}{l} \right) \right] - \frac{l}{2}(-\theta) = \frac{3}{16} \theta l + \frac{5}{16} a \end{aligned}$$

As can be seen, the calculation results of the above two approaches are the same.

5.9 Check for Calculated Results of Statically Indeterminate Structures

The calculation of internal forces is the basis of engineering structural design, and its accuracy is of vital importance. The calculation process of statically indeterminate structures is relatively complex with cumbersome mathematical computation. Therefore, after the internal forces are obtained, a check should be performed to ensure their correctness. From a global perspective, the check of internal forces includes whether the structural computational model is reasonable, whether the original data are correct, whether the selected parameters are appropriate, and whether the primary system is geometrically stable. The internal force calculation of

the structure is meaningful only if the computational model, the original data, and the selected parameters are correct.

At the same time, the check of internal forces requires using the basic concepts of mechanics, using the simplified estimation method, or performing qualitative analysis and judgment on the rationality of the calculation process and result based on relevant engineering experience. If no problem is found in these qualitative analyses, quantitative verification of the internal forces and internal force diagrams of the structure can be further performed according to the equilibrium conditions and deformation compatibility conditions.

Now we illustrate the checking process by taking the internal force diagrams of a statically indeterminate frame shown in figure 5.51a, b, and c as examples.

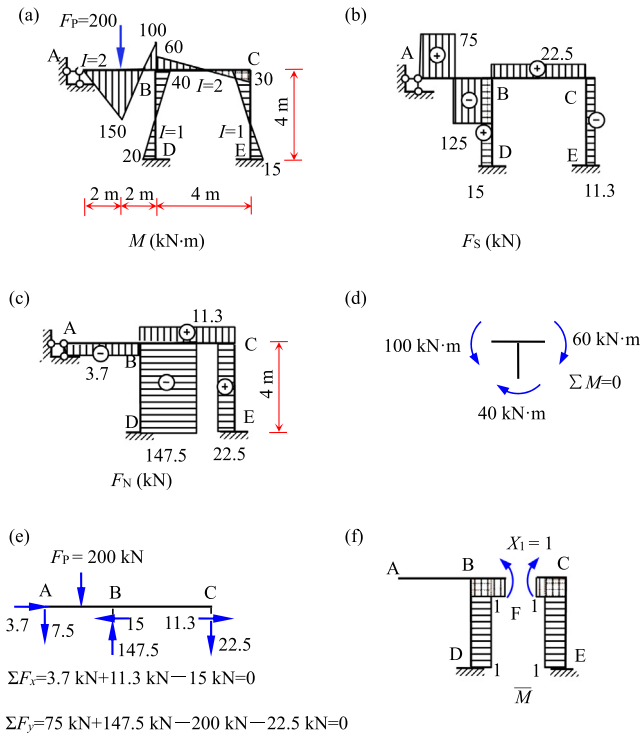


FIG. 5.51 – Check for calculated results of statically indeterminate frame.

5.9.1 Check of Equilibrium Conditions

Both the internal forces and reactions of statically determinate and indeterminate structures must meet the static equilibrium condition. The free body cut out from any part of the structure should satisfy the equilibrium equations. Generally, a node in a frame can be taken as the free body to check whether the equilibrium condition of the moment is met. The upper part above the cross-section of each column in a

frame can be taken as the free body to check whether the equilibrium condition of the force in the horizontal projection direction is met.

For example, in figure 5.51d, node B can be selected to check whether the equilibrium condition $\Sigma M = 0$ is satisfied; in figure 5.51e, member ABC can be selected to check whether the equilibrium conditions $\Sigma F_x = 0$, $\Sigma F_y = 0$ are satisfied. It can be seen from the figure that the above equilibrium conditions are all satisfied.

When checking the free body equilibrium conditions of a structure, generally only a few cases need to be checked. As long as any one of the free-body equilibrium conditions cannot be satisfied, it indicates an error in the calculation of the internal forces. In other words, it is a necessary condition for the correct calculation of internal forces that all the selected checking cases meet the free body equilibrium conditions.

For a statically indeterminate structure, there are infinite sets of internal forces that satisfy the equilibrium conditions. As the internal force diagram is obtained according to the equilibrium condition after the redundant constraint force is determined, the check of equilibrium condition is only valid for judging the correctness of the computation after the redundant constraint force is determined, but unable to judge whether the value of the redundant constraint force is correct. For this reason, it is also necessary to check the deformation conditions.

5.9.2 Check of Deformation Conditions

Since the redundant constraint forces are obtained from the deformation conditions, the correctness of the calculation can be checked by examining the deformation conditions. In fact, when calculating the displacement of a statically indeterminate structure, the virtual equilibrium state can be established on any of its corresponding primary structures. Therefore, the general process of checking the deformation condition is as follows: first, select an arbitrary primary structure and an arbitrary redundant unknown force X_i , calculate the displacement Δ_i in the direction of X_i according to the final internal force diagram, and then check whether Δ_i is equal to the corresponding displacement in the original structure (such as the given value a), that is:

$$\Delta_i = a \quad (5.19)$$

If the displacement Δ_i is calculated according to equation (5.18), then equation (5.19) becomes

$$\begin{aligned} & \Sigma \int \frac{\overline{M}M}{EI} ds + \Sigma \int \frac{\overline{F}_N F_N}{EA} ds + \Sigma \int \frac{k\overline{F}_S F_S}{GA} ds \\ & + \Sigma \int \frac{\overline{M}\alpha \Delta t}{h} ds + \Sigma \int \overline{F}_N \alpha t_0 ds - \Sigma \overline{F}_{RK} c_K = a \end{aligned} \quad (5.20)$$

where, $\overline{M}$, $\overline{F}_N$, $\overline{F}_S$ and $\overline{F}_R$ are the internal forces and support reaction of the primary structure under the unit load $X_i = 1$.

If the original structure is only subjected to loads, then

$$\Sigma \int \frac{\overline{M}M}{EI} ds + \Sigma \int \frac{\overline{F}_N F_N}{EA} ds + \Sigma \int \frac{k\overline{F}_S F_S}{GA} ds = 0 \quad (5.21)$$

It is worth noting that for a closed frame, the simplest check method is to use the deformation condition of zero relative rotation angle of any section on the closed frame to check the moment diagram. For example, in order to check the diagram M shown in figure 5.51a, the primary structure shown in figure 5.51f can be selected, and the moment of any section F in beam BC can be taken as the redundant unknown force X_1 . Then, under the action of unit load $X_1 = 1$, only the closed part DBCE of the frame generates a moment $\bar{M} = 1$. Therefore, the deformation condition (5.21) can be simplified to

$$\oint \frac{M}{EI} ds = 0 \quad (5.22)$$

From this, it is concluded that when a structure is only subjected to loads, the total area of the diagram (M/EI) along the closed frame should be zero.

Now we use this conclusion to check the diagram M in figure 5.51a. Integrating along path DBCE, the value is

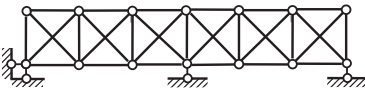
$$\begin{aligned} \oint \frac{M}{EI} ds &= \frac{1}{1} \times \left(-\frac{20 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} + \frac{40 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} \right) \\ &\quad + \frac{1}{2} \times \left(-\frac{60 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} + \frac{30 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} \right) \\ &\quad + \frac{1}{1} \times \left(-\frac{15 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} + \frac{30 \text{ kN} \cdot \text{m} \times 4 \text{ m}}{2} \right) \\ &= -130 \text{ kN} \cdot \text{m}^2 + 170 \text{ kN} \cdot \text{m}^2 \neq 0 \end{aligned}$$

It can be seen that the diagram M fails to meet the deformation condition, hence the calculated result is wrong.

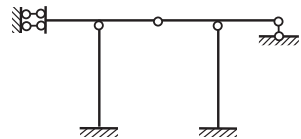
Problems

5.1 Determine the degree of indeterminacy of the structures shown in the following figure.

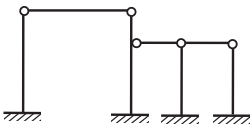
(a)



(b)



(c)



(d)

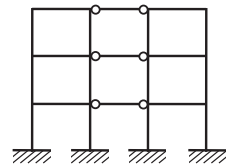
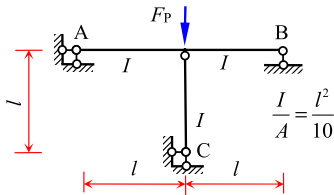


FIG. P5.1

5.2 Calculate the internal forces of the frames using the force method, and draw the diagram M .

(a)



(b)

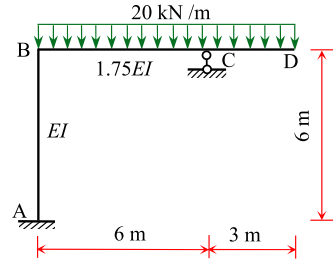
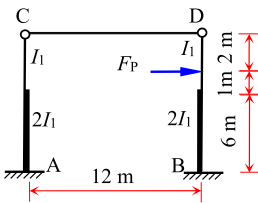


FIG. P5.2

5.3 Calculate the internal forces of the bents using the force method and draw the diagram M .

(a)



(b)

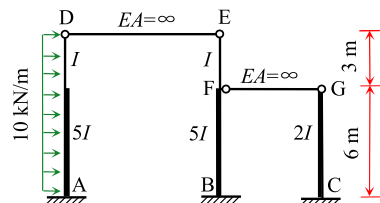
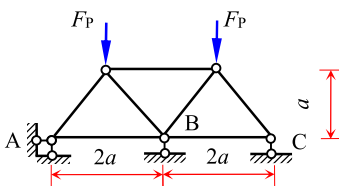


FIG. P5.3

5.4 Calculate the axial forces of the trusses shown in the following figure using the force method. For each member, $EA = \text{constant}$.

(a)



(b)

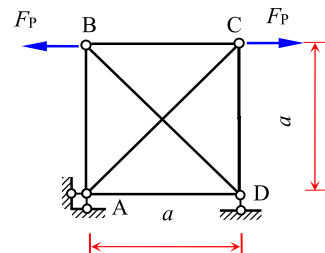


FIG. P5.4

5.5 As shown in the following figure, a continuous two-span suspended crane beam is subjected to a crane load of $F_P = 4.5 \text{ kN}$. Taking into account the axial deformation

of the booms, calculate the tensile force and elongation of the booms, and draw the diagrams M and F_S of the beam. The cross-sectional area of the $\Phi 20$ steel bar is $A = 3.14 \text{ cm}^2$, for the I20a steel beam, $I = 2370 \text{ cm}^4$.

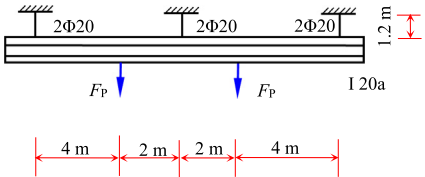


FIG. P5.5

5.6 Draw the diagram M of the symmetric frames shown in the following figure.

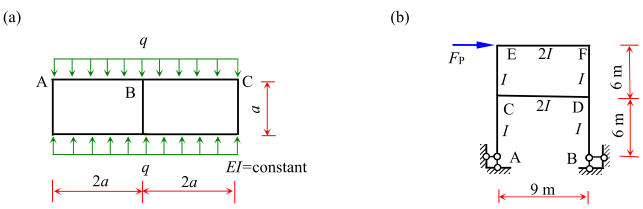


FIG. P5.6

5.7 In order to make the moment of section B of the beam shown in the following figure zero, how much should be the stiffness k of the elastic support? Then determine the deflection of point B under this condition.

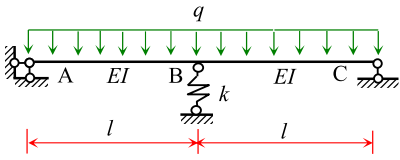


FIG. P5.7

5.8 Determine the horizontal thrust of the support of the semi-circular two-hinged arch with uniform cross-section, and draw the diagram M . Assume $EI = \text{constant}$ and consider only the influence of bending deformation on the displacement.

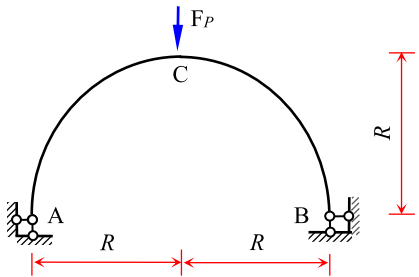


FIG. P5.8

5.9 Deduce the expression of the internal force of the parabolic two-hinged arch with tie bar under the uniformly distributed load. Assume EI of the arch section is constant and the arch axis equation is $y = \frac{4f}{l^2}x(l-x)$. When calculating the displacement, only the effect of moment in the arch needs to be considered and assume $ds = dx$.

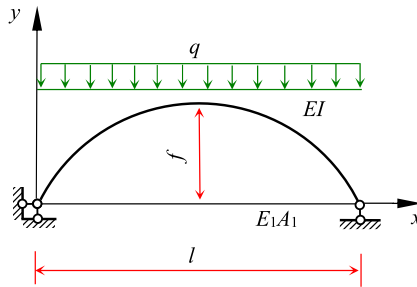


FIG. P5.9

5.10 Draw the moment diagram of the structure shown in the figure under the support movement. Assume EI is constant.

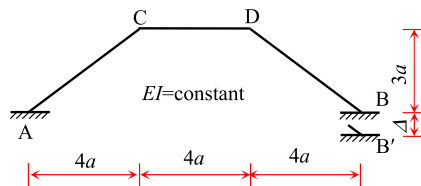


FIG. P5.10

5.11 Assume there is a rotational angle α at end A of the beams shown in the figure, draw the diagrams M and F_S of the beams. For each beam, select two primary systems for calculation, and determine the deflection curve equation and the maximum deflection of the beam.



FIG. P5.11

5.12 Assume there is a settlement of c at end B of the beam shown in the figure, and draw the diagrams M and F_S of the beam.

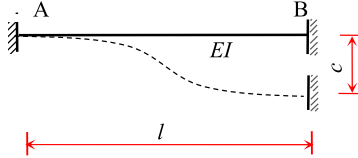


FIG. P5.12

5.13 For the beam shown in the figure, the temperature changes of the upper and lower sides are $+t_1$ and $+t_2$ ($t_2 > t_1$), respectively. The beam section has a height of h , and the thermal expansion coefficient is α . Draw the diagram M and determine the deflection curve equation.

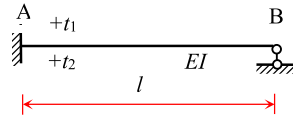


FIG. P5.13

5.14 For the portal frame shown in the figure, the inertia moment of the beam I_2 is s times that of column I_1 , i.e., $I_2 = sI_1$. There are three cases for the value of s : $s = 0.2, 1, 5$, and the roof height has four values: $f = 0$ m, 0.6 m, 2 m, 4 m. Determine the internal forces of the frame under the following conditions:

- (a) a fixed $f = 2$ m and various values of s .
- (b) a fixed $s = 1$ and various values of f .

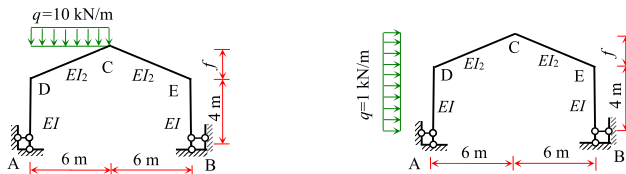


FIG. P5.14

Chapter 6

Displacement Method

The force method is the oldest basic method for the analysis of statically indeterminate structures, which is based on the analysis of statically determinate structures. The number of primary unknowns of the force method is just the degree of indeterminacy of the structure. With the development of modern industry, the redundancy of structures is increasing, and it is more and more difficult to solve the internal forces of these structures with many degrees of indeterminacy by the force method. Against this background, the **displacement method** for the analysis of multi-story and multi-span frame structures came into being.

The development of the displacement method can be divided into two stages: **the slope-deflection method** and **matrix displacement method**. In fact, in 1826, the French mechanician C. L. M. H. Navier (1785~1836) proposed the thought of the displacement method in elastic mechanics and used it to calculate the internal forces of statically indeterminate trusses. In 1914, Axel Bendixen, a Danish engineer, applied the slope-deflection method to the calculation of the internal forces of a frame with sidesway. In 1915, American scholars W. M. Wilson and G. A. Maney modified the secondary bending moment method, independently used it to solve the internal forces of frames, and named it the slope-deflection method. In 1926, the Danish scholar A. Ostenfeld established the canonical equation (namely, the basic equation) of the displacement method. In the 1950s, by using the computing power of computers, JH Argyris (Greece, 1913~2004) proposed a computer method for complex structural analysis—matrix displacement method, based on the combination of slope-deflection method and matrix algebra. Since then, for large-scale mechanical analysis of complex structures, traditional manual methods such as the force method and displacement method gradually stepped off the stage of history. Nevertheless, to understand and master the matrix displacement method and develop other new methods, it is still very important and necessary to study the basic analysis methods (force method and displacement method) of statically indeterminate structures.

The displacement method and force method are a pair of dual methods. Their solution process is similar in form. Both are based on the primary unknowns, primary system, and basic equation, but they differ in substance and form a duality, as shown in table 6.1. The force method takes the redundant constraint forces as the primary unknowns, and the basic equations of the force method (also called the flexibility

TAB. 6.1 – Comparison of force method and displacement method.

	Primary unknowns	Primary system	Basic equations	Scope of application
Force Method (proposed in 1886)	Redundant constraint forces	Statically determinate structure	Deformation compatibility conditions, flexibility equations	Statically indeterminate structures
Displacement method (proposed in 1915)	Independent nodal displacements	Assembled members	Static equilibrium conditions, stiffness equations	Statically determinate and indeterminate structures

equations) are established according to the deformation compatibility conditions. The redundant constraint forces and internal forces are solved for the analysis of statically indeterminate structures. The displacement method takes the independent joint displacements as the primary unknowns, and the basic equations of the displacement method (also called the stiffness equations) are established according to the static equilibrium conditions. The joint displacements and the internal forces of the member end are solved for the analysis of statically indeterminate structures. There are two ways to establish the basic equations of displacement method: one is to use the slope-deflection equation of bending member to directly establish the basic equations according to the joint moment equilibrium conditions and the sectional projection equilibrium equations, that is, the way of directly writing the equilibrium equations (*i.e.*, direct equilibrium method); and the other is to establish canonical equations (or basic equations) through the primary system and equilibrium conditions of displacement method, that is, the canonical equation method.

In this chapter, we will first give a detailed introduction to the basic concept, basic principle, and solution process of the first way of displacement method, as it is easier to understand. And then the second way of displacement method will be introduced: the primary system and basic equations of the displacement method. Finally, we will discuss the basic equations of displacement method derived from the principle of potential energy, the energy method for approximate calculation of structural displacement—the Rayleigh–Ritz method, and two commonly used asymptotic displacement methods: moment distribution method and no-shear distribution method.

6.1 Fundamental Concept of the Displacement Method

6.1.1 A Simple Example of Displacement Method

To understand the basic idea of the displacement method more specifically, a simple example of a truss is considered and shown in figure 6.1a.

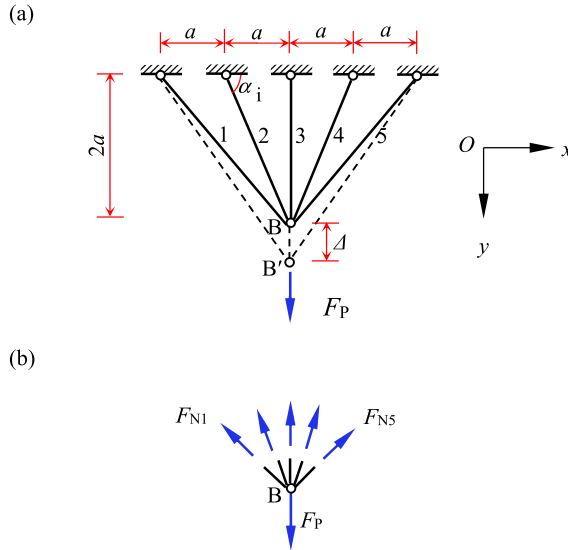


FIG. 6.1 – A symmetric truss.

Figure 6.1a is a symmetric truss structure subject to symmetric load F_P . Joint B only has a vertical displacement Δ , whose horizontal displacement is zero. If the force method is adopted and the symmetry of the structure is considered, the truss structure has two primary unknowns, *i.e.*, two redundant unknown forces. While for the displacement method, there is only one primary unknown: the vertical displacement Δ . This is because if the displacement Δ is known, the elongation deformation of each member can be obtained, and then the internal forces of each member can be calculated according to figure 6.1b, which describes the relationship between internal forces and external load, so the entire structural analysis problem can be solved. It can be seen that the displacement Δ is a critical unknown quantity.

How to solve the primary unknown Δ is the key to the structural analysis by displacement method, which can be divided into two steps.

The first step is to disassemble the structure, that is, take out a single member from the structure to perform analysis. Taking member AB in figure 6.2a for example, if the axial displacement of end B is known as u_i (*i.e.*, the elongation of member AB), the end force F_{Ni} can be written as

$$F_{Ni} = \frac{EA_i}{l_i} u_i \quad (6.1)$$

where E , A_i , l_i are the elastic modulus, cross-sectional area, and length of the member respectively. The coefficient $\frac{EA_i}{l_i}$ is the end force required to produce a unit displacement at the end of the member, which is called the stiffness coefficient of the axial force member. Equation (6.1) shows the relationship between the end force F_{Ni} and the end displacement u_i , which is called the stiffness equation of an axial force member.

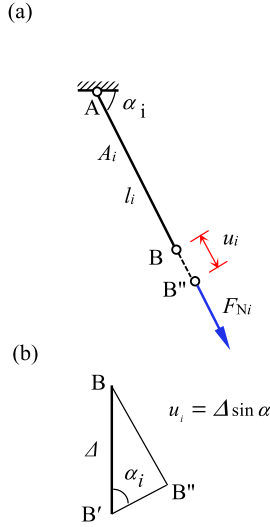


FIG. 6.2 – Typical member AB.

The second step is to assemble the structure, *i.e.*, assemble all members into the structure. When assembling, the vertical displacement of each member at the end B should be the same, because they all move from B to B'. This is the deformation compatibility condition. According to this deformation compatibility condition, in figure 6.2b the relationship between the end displacement u_i and the primary unknown Δ can be expressed as

$$u_i = \Delta \sin \alpha_i \quad (a)$$

Considering the equilibrium condition of joint B, $\sum F_y = 0$, we have (figure 6.1b)

$$\sum_{i=1}^5 F_{Ni} \sin \alpha_i = F_P \quad (b)$$

Substituting equation (6.1) and formula (a) into formula (b) yields

$$\sum_{i=1}^5 \frac{EA_i}{l_i} \sin^2 \alpha_i \Delta = F_P \quad (c)$$

This is the basic equation of the displacement method. It stands for the equilibrium equation, which is an equilibrium equation expressed by the displacement. From this equation, we can obtain the primary unknown.

$$\Delta = \frac{F_P}{\sum_{i=1}^5 \frac{EA_i}{l_i} \sin^2 \alpha_i} \quad (d)$$

So far, the key step for the calculation of the displacement method has been completed.

After the primary unknown Δ is obtained, the remaining problem can be easily solved. For example, in order to obtain the axial force of each member, substituting formula (d) into formula (a) and then into equation (6.1), then we have

$$F_{Ni} = \frac{\frac{EA_i}{l_i} \sin \alpha_i}{\sum_{i=1}^5 \frac{EA_i}{l_i} \sin^2 \alpha_i} F_P \quad (\text{e})$$

Assuming that the EA of each member is the same, according to the dimensions of each member in figure 6.1a and formulas (d) and (e), the primary unknown Δ and the internal forces of each member can be obtained as follows:

$$\Delta = 0.637 \frac{F_P}{EA}$$

$$F_{N1} = F_{N5} = 0.159 F_P$$

$$F_{N2} = F_{N4} = 0.255 F_P$$

$$F_{N3} = 0.319 F_P$$

From the above calculation process, it can be seen that the computational effort of the displacement method is not affected by the degree of indeterminacy of the structure. If the number of members is reduced from 5 to 2 in figure 6.1a, the truss structure becomes statically determinate, but the above method is still applicable. Therefore, the displacement method is independent of the degrees of indeterminacy of structures, which can be used to solve the internal forces of both statically determinate and statically indeterminate structures.

6.1.2 Primary Unknowns and Basic Equations of Displacement Method

According to the above simple example of the truss, the four main points of the displacement method can be summarized as follows:

- (1) **Primary unknowns of displacement method** are the independent joint displacements of the structure (*e.g.* the vertical displacement Δ of joint B in figure 6.1a).
- (2) **Basic equations of displacement method** are the equilibrium equations expressed by displacements (*e.g.* the vertical projection equilibrium equation (c) of joint B).
- (3) The process of establishing the basic equations can be divided into two steps: the first step is to disassemble the structure into members, analyze the members, and obtain the **stiffness equations** of the members; the second step is to assemble the members into the structure and carry out overall analysis of the structure to get the basic equations of displacement method.

This procedure is a process of first disassembly and then assembly. The primary unknown displacements can be solved from the basic equations of the displacement method. Finally, the displacements are replaced by the element stiffness equations to obtain the internal forces of the members. It can be seen that the basic idea of the displacement method is to transform the analysis of complex structures into the analysis of simple members.

- (4) The member analysis is the basis of structural analysis, and the stiffness equation of the member element is the basis of the basic equations of the displacement method. Therefore, the displacement method is also called the **stiffness method**.

6.1.3 Basic Idea of Calculating Frame Structures by Displacement Method

This chapter mainly discusses how to analyze and calculate the internal forces of statically indeterminate beams and frames by the displacement method. Now we will further describe the basic idea of the displacement method for frame structures. In the analysis of frames, usually, only the bending deformation is considered, while shear and tensile deformations are ignored, thus reducing the primary unknowns of the displacement method and still remaining high calculation accuracy.

Figure 6.3a shows a frame structure. As tensile deformation is neglected, joint A will have an angular displacement θ_A and a horizontal displacement Δ but no vertical displacement under the given load; and the horizontal displacement of joint C is the same as that of joint A. Because joint A is a rigid joint, when it has an angular displacement θ_A , the rotation of the end of each member connected to it should be equal to θ_A , according to the deformation condition. When the displacement method is used to calculate the internal forces of the frame, the independent joint displacements θ_A and Δ are taken as the primary unknowns. Once the primary unknowns, θ_A and Δ , are obtained, the calculation of the whole frame is transformed into the calculation of the members, as shown in figure 6.3b and c. The member AB is subjected to the known load q , in which end B is fixed and end A has known displacements θ_A and Δ . The member AC is subjected to the concentrated load F_P , in which end C is pinned and end A has a known displacement θ_A .

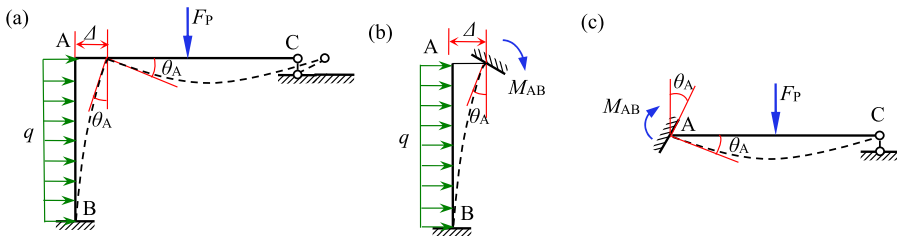


FIG. 6.3 – Primary unknowns of frame structure by using displacement method.

Therefore, when calculating a frame by the displacement method, the independent joint displacements are the critical primary unknowns. Once they are solved, the remaining problem is the calculation of the members. The basic idea of calculating a frame by displacement method is still disassembly and assembly. Firstly, the frame is disassembled into members, and the member element analysis is carried out, which is the calculation of the member under the known end displacement and the known load. Secondly, the members are rebuilt into the frame for overall analysis. The basic equation of the displacement method is established by using the equilibrium conditions of the frame, from which the primary unknowns can be obtained. Compared with truss structures, it is more complicated to solve the internal forces of frame structures, which will be discussed in detail later.

6.1.4 Determination of Primary Unknowns of Displacement Method

Like the force method, to use the displacement method, the number of primary unknowns should be first determined. The primary unknowns of the force method are the redundant constraint forces. The primary unknowns of the displacement method are the independent joint displacements, which can be categorized as angular displacements and linear displacements. In the force method, in order to simplify the calculation of statically indeterminate beams and frames, the deformations caused by axial forces and shear forces are usually neglected in the calculation of flexibility coefficients and free terms. In the displacement method, each rigid joint of a plane frame has one angular displacement and two linear displacements (horizontal and vertical), totaling three independent displacements. To reduce the number of primary unknowns and simplify the calculation, the axial deformation of bending members is usually ignored and the bending deformation is assumed to be small. Therefore, it is assumed that the distance between the two ends of a bending member remains unchanged after deformation, so that each bending member is equivalent to a link constraint, thus reducing the number of independent joint linear displacements in frame structures with sidesway. Now we will illustrate how to determine the number of angular displacements and linear displacements in the analysis of frames by the displacement method.

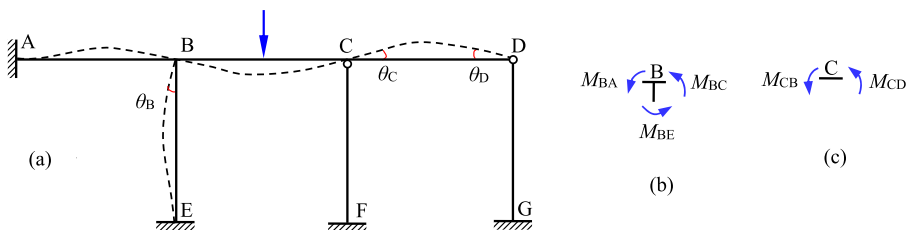


FIG. 6.4 – A statically indeterminate frame without sidesway.

Figure 6.4a illustrates a statically indeterminate frame without sidesway. Under the external force shown in the figure, assuming that the axial force is negligible and the bending deformation is small, the deformation of the structure is shown by the dashed line in the figure. At rigid joint B, the deformation compatibility condition gives that the rotation angles of the three members connected to joint B are the same. So there is only one independent angular displacement θ_B at rigid joint B. Similarly, at rigid joint C, there is only one independent angular displacement θ_C . There is also an unknown angular displacement θ_D at pinned joint D.

Since the end moments of each member at the rigid joint are generally not zero, as shown in figure 6.4b and c, the moment equilibrium conditions can be written as follows at joints B and C.

$$\left. \begin{aligned} M_{BC} + M_{BA} + M_{BE} &= 0 \\ M_{CB} + M_{CD} &= 0 \end{aligned} \right\} \quad (a)$$

Since joint D is pinned, the member end moment is equal to zero. And the moment equilibrium condition at joint D is an equation of zero equal to zero. Only two unknowns need to be resolved in the two equilibrium conditions of formula (a), so the primary unknowns in this problem are the angular displacement θ_B and θ_C at joints B and C. Although θ_D is unknown, it is a non-independent joint displacement that cannot be regarded as a primary unknown. Therefore, in the equilibrium conditions or basic equation (a), the end moments of each member should be expressed as functions of θ_B and θ_C , which will be discussed in the next section.

Through the above analysis, it can be seen that in the displacement method, it is unnecessary to treat all joint displacements as primary unknowns. The angular displacement of joint D in figure 6.4a is not an independent joint displacement, thus it is unnecessary to regard it as a primary unknown. The number of independent angular displacements in the displacement method is equal to the number of rotatable rigid joints in the structure.

Now, the frame with sidesway in figure 6.5a is taken as an example to discuss the problem of determining the number of linear displacements. As the distance between two ends of any member is assumed to be constant, in the case of small displacement, joints C and D have no vertical displacement, and their horizontal displacements are identical, which can be expressed by a symbol Δ . Therefore, the four linear displacements of the two joints are now reduced to a single linear displacement Δ . There are only three primary unknowns, namely, θ_C , θ_D , and Δ .

For a general frame, the number of independent linear displacements of joints can often be determined by observation. Figures 6.5b and c demonstrate two examples of bent and frame, in which the dashed line represents the curve of deformed members. In figure 6.5b, there is only one independent linear displacement Δ , because joints D, E, and F are connected by horizontal beams, and the horizontal linear displacement must be the same when the tensile deformation is neglected. Figure 6.5c shows a two-story frame composed of horizontal beams and columns. Four rigid joints C, D, E, and F have four angular displacements. In addition, there are two independent joint linear displacements Δ_1 and Δ_2 . Obviously, each story has a linear displacement, and thus the number of independent linear displacements of joints is equal to the number of stories in the frame.

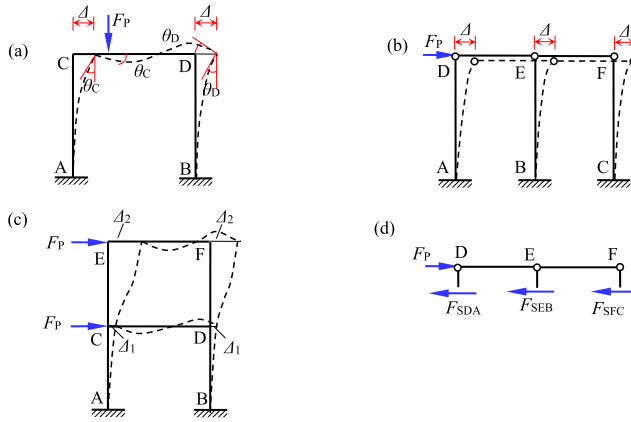


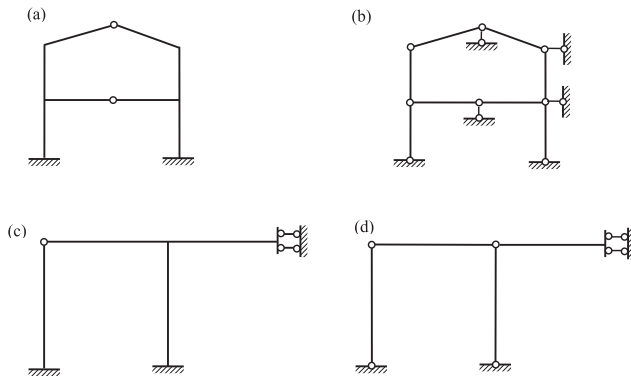
FIG. 6.5 – Primary unknowns of displacement method for frame and bent.

For the linear displacement Δ shown in figure 6.5b, the partial structure in figure 6.5d can be taken to establish the sectional projection equilibrium equation of horizontal forces, that is:

$$F_{SDA} + F_{SEB} + F_{SFC} - F_P = 0 \quad (b)$$

Formula (b) is the basic equation of the displacement method for the structure in figure 6.5b. F_{SDA} , F_{SEB} , and F_{SFC} should be expressed as a function of the primary unknown Δ . The functional relationship between the displacements and internal forces at the member ends will be discussed in the next section.

For more complicated frames that are difficult to determine the independent linear displacements by observation, the method of geometric construction analysis can be used.

FIG. 6.6 – Determination of independent linear displacements of two frames *via* geometric construction analysis.

Taking the frame shown in figure 6.6a as an example, since each joint has two linear displacements, and each straight member provides a constraint condition with a constant distance between two ends, this is similar to the rule of geometric construction analysis of the planar hinged system in chapter 2 (each joint of the planar hinged system has two degrees of freedom, and each link has one constraint). To determine the number of independent joint linear displacements, all rigid joints and fixed supports in the frame can be transformed into pinned joints and pin supports, so as to form the hinged system shown in figure 6.6b. After adding four link constraints, the hinged system can be transformed from a geometrically changeable system to an unchangeable system. Therefore, the frame in figure 6.6a has four independent joint linear displacements. Similarly, all the rigid joints in figure 6.6c can be transformed into pinned joints, and the hinged system shown in figure 6.6d is obtained. The system is geometrically unchangeable, so the frame in figure 6.6c has no linear displacement. In summary, the number of independent joint linear displacements in the displacement method is equal to the degrees of freedom of the hinged system after all the rigid joints (including fixed supports) in the structure are transformed into pinned joints, or the number of links that need to be added to make the hinged system geometrically unchangeable.

In general, when the displacement method is used to calculate statically indeterminate frames, the primary unknowns include independent joint angular and linear displacements. The number of independent angular displacements is equal to the number of rotatable rigid joints, and the number of independent linear displacements is equal to the degrees of freedom of the hinged system. The number of primary unknowns in the displacement method is the number of kinematic indeterminacy, while the number of primary unknowns in the force method is that of static indeterminacy. Selecting the primary unknowns of the displacement method, not only ensures that the member end rotation angles at rigid joints are identical but also ensures that the distance at the two ends of each member remains unchanged. Therefore, in the process of disassembling and assembling, the compatibility condition of displacement of each member can be ensured, thus the continuous deformation condition can be satisfied. Correspondingly, the basic equations of the displacement method are the equilibrium equations expressed by the angular and linear displacements, including the moment equilibrium equations of joints and the force projection equilibrium equations of sections.

6.2 Slope-Deflection Equation of Prismatic Member

To use the displacement method to calculate frame structures, the physical relationship between the forces and displacements of each member end needs to be established. In this section, two problems in the calculation of prismatic members with constant cross-section will be discussed: one is to calculate the member end moments under known displacements; the other is to calculate the fixed-end moments under known load.

In figure 6.7a, a frame structure is deformed as shown by the dashed line under the load. Among all members, the ends of beam AB have not only rotational angles:

θ_A and θ_B , but also a relative linear displacement, named the end translation Δ . Compared with other members in the structure, member AB is called a general bending member.

The deformed member AB is shown in figure 6.7b. The member end moments M_{AB} , M_{BA} , and shears F_{SAB} , F_{SBA} are collectively called the member end forces. The rotational angles θ_A , θ_B , and relative translation Δ between the two ends of the member in the direction perpendicular to the original axis, are collectively referred to as the member end displacements. Under the premise of small deformation, if the length of member AB is l , the chord rotation angle can be obtained from Δ , *i.e.*, $\varphi = \Delta/l$.

For the convenience of operation, the following sign conventions are adopted in the displacement method: the end rotations θ_A , θ_B , the chord rotation φ , and the end moments M_{AB} , M_{BA} , are all positive in the clockwise direction. The sign of end shears remains the same as the previous convention. It is worth noting that the sign rules about the member end moments here are different from the usual sign convention of the sectional moments (for example, the moment in a beam is specified as positive if it makes the lower part of the beam tensile). Firstly, the sign conventions here is for the member end moment, not for the moment of any section of the member; secondly, when the member (or joint) is taken as the free body, the member end moments are the external forces on the free body.

Hence, to facilitate the establishment of the equilibrium equation of the free body (the basic equation of displacement method), the end moments are always positive in the clockwise direction and are treated as the external forces of the member in this chapter. On the other hand, when drawing a moment diagram, the moment is regarded as the internal force of the member, so the usual sign convention should still be followed. In short, the member end moment has dual identity: it is both the internal force of the member and the external force of the free body. We should pay attention to using the corresponding sign rules in different situations.

In figure 6.7b, all the quantities are set as positive. In order to express the end forces of member AB as a function of the member end displacements and external loads, the pre-deformed member AB can be regarded as a single-span statically indeterminate beam fixed at both ends. The force and deformation properties of the simply supported beam are the same as the original statically indeterminate beam fixed at both ends. Then, the end forces of member AB can be solved in three steps:

Firstly, let the member end displacements act on the single-span statically indeterminate beam to generate the deformation shown in figure 6.7c, and generate the member end moments M'_{AB} , M'_{BA} , and the member end shears F'_{SAB} , F'_{SBA} , so as to establish the formula of the end forces from the end displacements.

Secondly, let the load F_P act on the single-span statically indeterminate beam alone to generate the deformation shown in figure 6.7d, and generate the fixed-end moments M^F_{AB} , M^F_{BA} and fixed-end shears F^F_{SAB} , F^F_{SBA} . These fixed-end forces can be obtained by the force method.

Thirdly, add up the member end forces in figure 6.7c and the fixed-end forces in figure 6.7d to obtain the total member end forces under the combined action of θ_A , θ_B , Δ , and F_P in figure 6.7b, that is

$$\left. \begin{aligned} M_{AB} &= M'_{AB} + M_{AB}^F \\ M_{BA} &= M'_{BA} + M_{BA}^F \\ F_{SAB} &= F'_{SAB} + F_{SAB}^F \\ F_{SBA} &= F'_{SBA} + F_{SBA}^F \end{aligned} \right\} \quad (a)$$

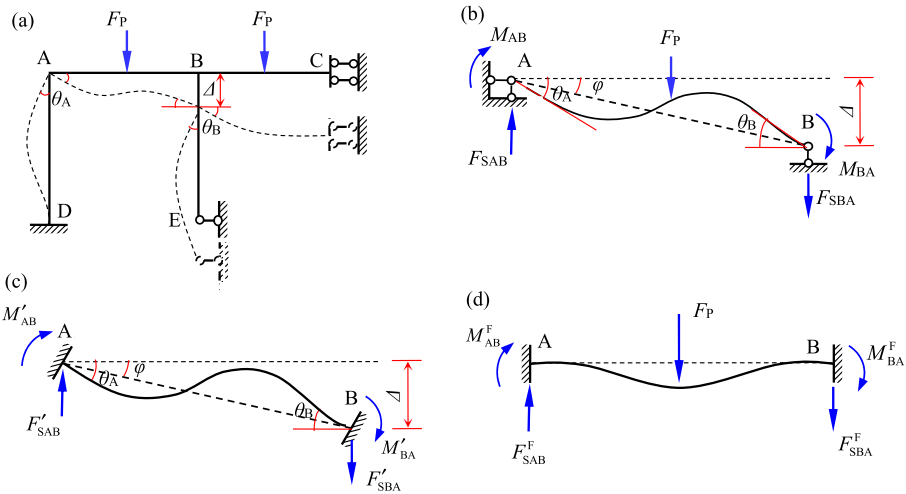


FIG. 6.7 – (a) Typical members of frame, (b) simply supported beam under load and member end displacements, (c) statically indeterminate beam under member end displacements, and (d) statically indeterminate beam under load.

6.2.1 Calculating Member-End Internal Forces from Member-End Displacements

Now we discuss the first step: how to determine the member end forces from the member end displacements. Figure 6.8 shows a straight member AB with a constant cross-section (the moment of inertia I is constant). Given that the rotations of the ends A and B are θ_A and θ_B , respectively, and the relative displacement between the two ends of the member in the direction perpendicular to the undeformed axis of the member is Δ . The end moments of M_{AB} and M_{BA} are to be determined. It should be pointed out that if the member moves in parallel or perpendicular to the member axis, there will be no moment at the member ends. Therefore, it is only necessary to consider the situation when the relative displacement Δ of the two ends in the direction perpendicular to the member axis exists. In addition, the chord rotation can be derived from Δ , namely, $\varphi = \Delta/l$.

Firstly, the end rotational angles of the simply supported beam under the end couple moments M_{AB} and M_{BA} should be determined (figure 6.9a), which can be achieved by the unit load method,

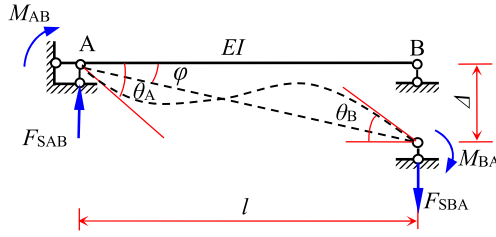
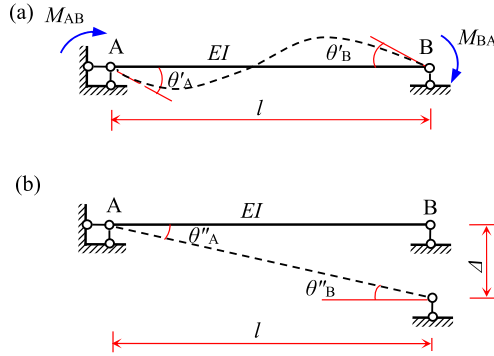


FIG. 6.8 – A straight member AB with constant cross-section.

FIG. 6.9 – Straight member AB under (a) end couple moments M_{AB} and M_{BA} and (b) relative vertical displacement Δ .

$$\left. \begin{aligned} \theta'_A &= \frac{1}{3i} M_{AB} - \frac{1}{6i} M_{BA} \\ \theta'_B &= -\frac{1}{6i} M_{AB} + \frac{1}{3i} M_{BA} \end{aligned} \right\} \quad (6.2)$$

where, $i = \frac{EI}{l}$ is called the linear stiffness of the member.

Secondly, when there is a relative vertical displacement Δ between the two ends of the simply supported beam (figure 6.9b), the end rotational angles can be written as

$$\theta''_A = \theta''_B = \frac{\Delta}{l} \quad (6.3)$$

In summary, the two ends have moments M_{AB} and M_{BA} , and there is a relative vertical displacement Δ between the two ends. The end rotational angles are

$$\left. \begin{aligned} \theta_A &= \frac{1}{3i} M_{AB} - \frac{1}{6i} M_{BA} + \frac{\Delta}{l} \\ \theta_B &= -\frac{1}{6i} M_{AB} + \frac{1}{3i} M_{BA} + \frac{\Delta}{l} \end{aligned} \right\} \quad (6.4)$$

Solving the above equations yields

$$\left. \begin{aligned} M_{AB} &= 4i\theta_A + 2i\theta_B - 6i\frac{\Delta}{l} \\ M_{BA} &= 2i\theta_A + 4i\theta_B - 6i\frac{\Delta}{l} \end{aligned} \right\} \quad (6.5)$$

Equation (6.5) is the equation for calculating end moments from end displacements θ_A , θ_B , and Δ (it is customarily called the slope-deflection equation). This equation can also be obtained by solving the straight member with a constant cross-section fixed at both ends under support displacements by the force method. The concerned end is usually called the near-end and the other one is called the far-end. It can be seen from the above equation that end moments are caused by three parts: the near-end rotation, the far-end rotation, and the chord rotation, which can be summarized as a pithy formula of “near 4, far 2, chord -6”. In addition, the end shears can also be obtained from the equilibrium conditions as follows:

$$F_{SAB} = F_{SBA} = -\frac{1}{l}(M_{AB} + M_{BA})$$

Substituting equation (6.5) into it, we have

$$F_{SAB} = F_{SBA} = -\frac{6i}{l}\theta_A - \frac{6i}{l}\theta_B + \frac{12i}{l^2}\Delta \quad (6.6)$$

For the sake of compactness, equations (6.5) and (6.6) can be written in matrix form:

$$\begin{bmatrix} M_{AB} \\ M_{BA} \\ F_{SAB} \end{bmatrix} = \begin{bmatrix} 4i & 2i & -\frac{6i}{l} \\ 2i & 4i & -\frac{6i}{l} \\ -\frac{6i}{l} & -\frac{6i}{l} & \frac{12i}{l^2} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \Delta \end{bmatrix} \quad (6.7)$$

Equation (6.7) has similar properties to equation (6.1) and it is called the stiffness equation of the bending member. In equation (6.7), the matrix

$$\begin{bmatrix} 4i & 2i & -\frac{6i}{l} \\ 2i & 4i & -\frac{6i}{l} \\ -\frac{6i}{l} & -\frac{6i}{l} & \frac{12i}{l^2} \end{bmatrix}$$

is called the **stiffness matrix** of the bending member, and the coefficients are called **stiffness coefficients**. The stiffness coefficients are constants only related to the length, cross-sectional shape, and dimension as well as material properties of the member. They are also called shape constants by convention.

Next, we will discuss the slope deflection equations of a member with end A fixed and end B constrained by three different common supports.

- (1) End B is a fixed support (figure 6.10a).

Let $\theta_B = 0$ in equation (6.5), we get

$$\left. \begin{aligned} M_{AB} &= 4i\theta_A - 6i\frac{\Delta}{l} \\ M_{BA} &= 2i\theta_A - 6i\frac{\Delta}{l} \end{aligned} \right\} \quad (6.8)$$

- (2) End B is pinned (figure 6.10b).

In the second formula of equation (6.5), let $M_{BA} = 0$, we get

$$\theta_B = \frac{3}{2}\frac{\Delta}{l} - \frac{1}{2}\theta_A$$

It can be seen that the rotational displacement at the pin support is not independent. Substituting it into the first formula of equation (6.5), we get

$$M_{AB} = 3i\theta_A - 3i\frac{\Delta}{l} \quad (6.9)$$

- (3) End B is a slider support (figure 6.10c).

Let $\theta_B = 0$ and $F_{SAB} = F_{SBA} = 0$ in equation (6.6), we get

$$\frac{\Delta}{l} = \frac{1}{2}\theta_A$$

It can be seen that the relative translation of the slider support is not independent. Substituting it into equation (6.5) yields

$$\left. \begin{aligned} M_{AB} &= i\theta_A \\ M_{BA} &= -i\theta_A \end{aligned} \right\} \quad (6.10)$$

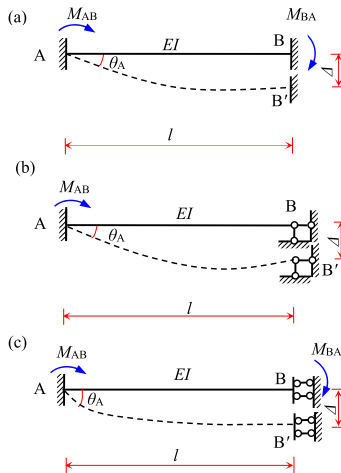


FIG. 6.10 – Three typical members with end B constrained by different supports.

6.2.2 Calculating Fixed-End Internal Forces from Loads

There are three kinds of representative bending members with constant cross-section: (1) a beam fixed at both ends; (2) a beam fixed at one end and pinned at the other end; (3) a beam fixed at one end and slider-supported at the other end. Table 6.2 lists the end moments and the end shears of three kinds of bending members under common loads, which are called the **fixed-end moments** and the **fixed-end shears**. Because they are constants only related to the loads, they are also called the load constants. The fixed-end moments are expressed by M_{AB}^F and M_{BA}^F , and the fixed-end shears are expressed by F_{SAB}^F and F_{SBA}^F .

For a beam fixed at both ends, the third formula in table 6.2 is the basic formula. By using this formula and the superposition principle, the fixed-end moments under the combined action of the concentrated force F_{Pi} and distributed load $q(a)$ can be written as follows (figure 6.11a and b):

$$\left. \begin{aligned} M_{AB}^F &= -\sum F_{Pi} \frac{a_i(l-a_i)^2}{l^2} - \int_0^l \frac{q(a)a(l-a)^2}{l^2} da \\ M_{BA}^F &= \sum F_{Pi} \frac{a_i^2(l-a_i)}{l^2} - \int_0^l \frac{q(a)a^2(l-a)}{l^2} da \end{aligned} \right\} \quad (6.11)$$

where, in figure 6.11a a_i stands for the distance between concentrated load F_{Pi} and the end A, and in figure 6.11b a denotes the distance between the load $q(a)$ da and the end A.

If the beam with constant cross-section is subjected to known loads and known member end displacements, according to the superposition principle, the general formula of the member end moments (refer to equation (6.5)) can be written as

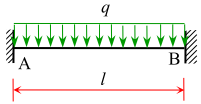
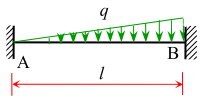
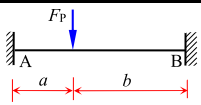
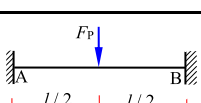
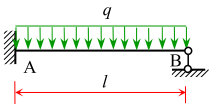
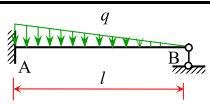
$$\left. \begin{aligned} M_{AB} &= 4i\theta_A + 2i\theta_B - 6i\frac{\Delta}{l} + M_{AB}^F \\ M_{BA} &= 2i\theta_A + 4i\theta_B - 6i\frac{\Delta}{l} + M_{BA}^F \end{aligned} \right\} \quad (6.12)$$

The general formula of the member end shears (refer to equation (6.6)) can be written as

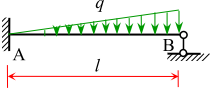
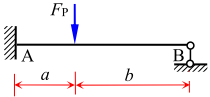
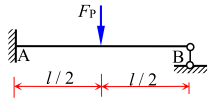
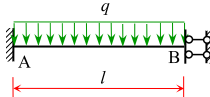
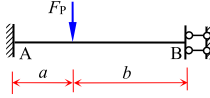
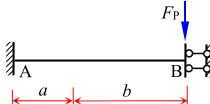
$$\left. \begin{aligned} F_{SAB} &= -\frac{6i}{l}\theta_A - \frac{6i}{l}\theta_B + \frac{12i}{l^2}\Delta + F_{SAB}^F \\ F_{SBA} &= -\frac{6i}{l}\theta_A - \frac{6i}{l}\theta_B + \frac{12i}{l^2}\Delta + F_{SBA}^F \end{aligned} \right\} \quad (6.13)$$

In addition, the fixed-end moments of three types of beams are interrelated. For example, the fixed-end moments of the second and third-type beams can be derived from the first-type beam.

TAB. 6.2 – Fixed-end moments and fixed-end shears of members with constant cross-section.

	No	Diagram of beams under loads	Fixed-end moments (positive in clockwise direction)	Fixed-end shears
Two ends fixed	1		$M_{AB}^F = -\frac{ql^2}{12}$ $M_{BA}^F = \frac{ql^2}{12}$	$F_{SAB}^F = \frac{ql}{2}$ $F_{SBA}^F = -\frac{ql}{2}$
	2		$M_{AB}^F = -\frac{ql^2}{30}$ $M_{BA}^F = \frac{ql^2}{20}$	$F_{SAB}^F = \frac{3ql}{20}$ $F_{SBA}^F = -\frac{7ql}{20}$
	3		$M_{AB}^F = -\frac{F_P ab^2}{l^2}$ $M_{BA}^F = \frac{F_P a^2 b}{l^2}$	$F_{SAB}^F = \frac{F_P b^2}{l^2} \left(1 + \frac{2a}{l}\right)$ $F_{SBA}^F = -\frac{F_P a^2}{l^2} \left(1 + \frac{2b}{l}\right)$
	4		$M_{AB}^F = -\frac{F_P l}{8}$ $M_{BA}^F = \frac{F_P l}{8}$	$F_{SAB}^F = \frac{F_P}{2}$ $F_{SBA}^F = -\frac{F_P}{2}$
One end fixed, the other end pinned	5		$M_{AB}^F = -\frac{ql^2}{8}$	$F_{SAB}^F = \frac{5}{8} ql$ $F_{SBA}^F = -\frac{3}{8} ql$
	6		$M_{AB}^F = -\frac{ql^2}{15}$	$F_{SAB}^F = \frac{2}{5} ql$ $F_{SBA}^F = -\frac{1}{10} ql$

TAB. 6.2 – (continued).

	No	Diagram of beams under loads	Fixed-end moments (positive in clockwise direction)	Fixed-end shears
	7		$M_{AB}^F = -\frac{7ql^2}{120}$	$F_{SAB}^F = \frac{9}{40}ql$ $F_{SBA}^F = -\frac{11}{40}ql$
	8		$M_{AB}^F = -\frac{F_P b(l^2 - b^2)}{2l^2}$	$F_{SAB}^F = \frac{F_P b(3l^2 - b^2)}{2l^3}$ $F_{SBA}^F = \frac{F_P a^2(3l - a)}{2l^3}$
	9		$M_{AB}^F = -\frac{3F_P l}{16}$	$F_{SAB}^F = \frac{11}{16}F_P$ $F_{SBA}^F = -\frac{5}{16}F_P$
	10		$M_{AB}^F = -\frac{ql^2}{3}$ $M_{BA}^F = -\frac{ql^2}{6}$	$F_{SAB}^F = ql$ $F_{SBA}^F = 0$
	11		$M_{AB}^F = -\frac{F_P a}{2l}(2l - a)$ $M_{BA}^F = -\frac{F_P a^2}{2l}$	$F_{SAB}^F = F_P$ $F_{SBA}^F = 0$
	12		$M_{AB}^F = M_{BA}^F = -\frac{F_P l}{2}$	$F_{SAB}^F = F_P$ $F_{SBA}^{LF} = F_P$ $F_{SBA}^{RF} = 0$
		One end fixed, the other end slider-supported		

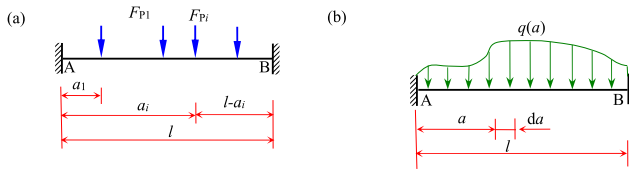


FIG. 6.11 – Beam with two fixed supports under (a) concentrated loads and (b) distributed load.

6.3 Analysis of Frames Without Sidesway

If the joints of a frame (excluding the supports and the boundary joints) have only angular displacements but no independent linear displacements, this frame is called a **frame without sidesway**.

This section discusses the analysis and calculation of a frame without sidesway by the slope deflection equation of members. The calculation of continuous beams also belongs to this type of problem.

6.3.1 Selection of Primary Unknowns

Figure 6.12a shows a continuous beam. Under the given loads, joint B only has the angular displacement θ_B and no linear displacement. It is a problem of beam without sidesway.

When the displacement method is used to calculate the frame, the **angular displacement of joint θ_B** is taken as the primary unknown. It should be noted that though there is an angular displacement at the pin support C, it is not an independent displacement and is not selected as a primary unknown.

The fixed-end moments of each member can be determined from table 6.2:

$$\begin{aligned} -M_{AB}^F &= M_{BA}^F = \frac{20 \text{ kN} \times 6 \text{ m}}{8} = 15 \text{ kN} \cdot \text{m} \\ M_{BC}^F &= -\frac{2 \text{ kN/m} \times (6 \text{ m})^2}{8} = -9 \text{ kN} \cdot \text{m} \end{aligned}$$

By using the slope-deflection equations (6.8) and (6.9) (in which, $\Delta = 0$), the member end moments can be expressed as follows (assuming the line stiffness i is identical):

$$\left. \begin{aligned} M_{AB} &= 2i\theta_B - 15 \text{ kN} \cdot \text{m} \\ M_{BA} &= 4i\theta_B + 15 \text{ kN} \cdot \text{m} \\ M_{BC} &= 3i\theta_B - 9 \text{ kN} \cdot \text{m} \end{aligned} \right\} \quad (\text{a})$$

Therefore, once θ_B is determined, the member end moments can be obtained.

6.3.2 Establishment of Basic Equations

In order to solve the primary unknown θ_B , the basic equation of the displacement method should be established. Taking joint B as a free body (figure 6.12b), the moment equilibrium equation can be written as:

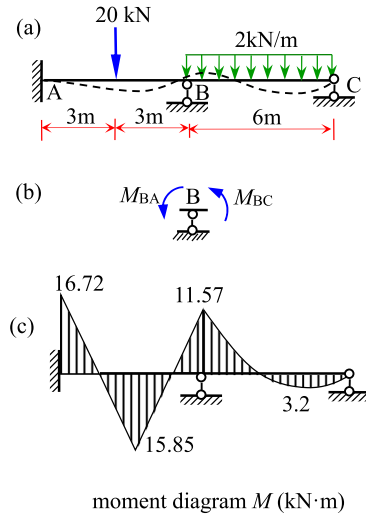


FIG. 6.12 – Analysis of continuous beam.

$$\sum M_B = 0, M_{BA} + M_{BC} = 0 \quad (b)$$

By using formula (a), this equilibrium equation can be written as

$$7i\theta_B + 6 \text{ kN}\cdot\text{m} = 0 \quad (c)$$

Formula (c) is the equilibrium equation expressed by θ_B , that is, the basic equation of displacement method, from which the primary unknown can be obtained.

$$\theta_B = -\frac{6 \text{ kN}\cdot\text{m}}{7i} \quad (d)$$

So far, the key problem of displacement method has been solved. The end moments of each member can be determined by substituting formula (d) into formula (a),

$$M_{AB} = 2i \times \left(-\frac{6 \text{ kN}\cdot\text{m}}{7i} \right) - 15 \text{ kN}\cdot\text{m} = -16.72 \text{ kN}\cdot\text{m}$$

$$M_{BA} = 4i \times \left(-\frac{6 \text{ kN}\cdot\text{m}}{7i} \right) + 15 \text{ kN}\cdot\text{m} = 11.57 \text{ kN}\cdot\text{m}$$

$$M_{BC} = 3i \times \left(-\frac{6 \text{ kN}\cdot\text{m}}{7i} \right) - 9 \text{ kN}\cdot\text{m} = -11.57 \text{ kN}\cdot\text{m}$$

Then, the moment diagram can be drawn, as shown in figure 6.12c.

In general, when using the displacement method to solve the statically indeterminate continuous beams and frames without sidesway, there is a joint rotation at each rigid joint – a primary unknown. Correspondingly, at each rigid joint, we can also write a moment equilibrium equation – the basic equation. Therefore, the

number of basic equations is exactly equal to the number of primary unknowns. And then all primary unknowns can be solved.

The basic idea of displacement method is to first disassemble and then assemble. There are two principles for assembly: firstly, the deformation of each member at the joint should be compatible; secondly, the assembled joints must meet the equilibrium conditions. The first requirement, it has been taken into account when selecting the primary unknowns of displacement method. Since there is only one angular displacement at each joint, that is, the end rotational displacement of each member at the rigid joint is specified to be identical, and the **continuity condition of deformation** at the joint can be ensured. For the second requirement, it is only considered when the basic equations of displacement method are established, as the basic equations are written according to the equilibrium conditions of the joints. Therefore, the solution of the displacement method has satisfied the equilibrium conditions and the continuity condition of deformation, and we know how the two conditions are satisfied.

Example 6.1. Draw the moment diagram of the frame shown in figure 6.13a.

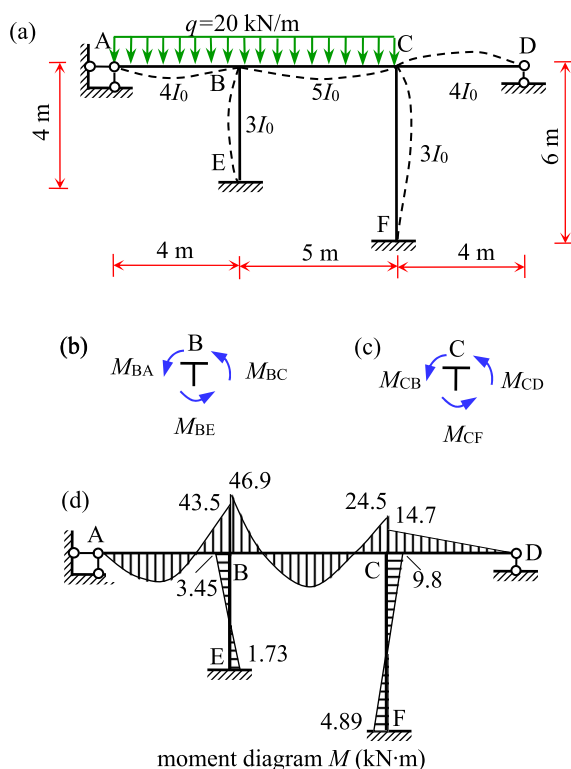


FIG. 6.13 – Analysis of frame structure without sidesway.

Solution: (1) Determine the primary unknowns.

There are two primary unknowns in all: θ_B , θ_C .

(2) Calculate the member end moments.

The fixed-end moments can be obtained from table 6.2.

$$\begin{aligned} M_{BA}^F &= \frac{ql^2}{8} = \frac{20 \text{ kN/m} \times (4 \text{ m})^2}{8} \\ &= 40.0 \text{ kN}\cdot\text{m} \end{aligned}$$

$$\begin{aligned} M_{BC}^F &= -\frac{ql^2}{12} = -\frac{20 \text{ kN/m} \times (5 \text{ m})^2}{12} \\ &= -41.7 \text{ kN}\cdot\text{m} \end{aligned}$$

$$M_{CB}^F = 41.7 \text{ kN}\cdot\text{m}$$

The stiffnesses of frame members are taken as relative values for calculation. Assuming $EI_0 = 1$, we have

$$i_{AB} = \frac{4EI_0}{4} = 1, \quad i_{BC} = \frac{5EI_0}{5} = 1, \quad i_{CD} = \frac{4EI_0}{4} = 1, \quad i_{BE} = \frac{3EI_0}{4} = \frac{3}{4}, \quad i_{CF} = \frac{3EI_0}{6} = \frac{1}{2}$$

According to the slope-deflection equations (6.5), (6.8), and (6.9), the end moments of each member can be written as follows, after superimposing the fixed-end moment:

$$M_{BA} = 3i_{AB}\theta_B + M_{BA}^F = 3\theta_B + 40.0$$

$$M_{BC} = 4i_{BC}\theta_B + 2i_{BC}\theta_C + M_{BC}^F = 4\theta_B + 2\theta_C - 41.7$$

$$M_{CB} = 2i_{BC}\theta_B + 4i_{BC}\theta_C + M_{CB}^F = 2\theta_B + 4\theta_C + 41.7$$

$$M_{CD} = 3i_{CD}\theta_C = 3\theta_C$$

$$M_{BE} = 4i_{BE}\theta_B = 3\theta_B, \quad M_{EB} = 2i_{BE}\theta_B = 1.5\theta_B$$

$$M_{CF} = 4i_{CF}\theta_C = 2\theta_C, \quad M_{FC} = 2i_{CF}\theta_C = \theta_C$$

(3) Establish the displacement method equations.

According to the equilibrium equation of joint B (figure 6.13b):

$$\sum M_B = 0, \quad M_{BA} + M_{BC} + M_{BE} = 0$$

Substituting the results of the previous step into it yields

$$10\theta_B + 20\theta_C - 1.7 = 0 \quad (a)$$

According to the equilibrium equation of joint C (figure 6.13c)

$$\sum M_C = 0, M_{CB} + M_{CD} + M_{CF} = 0$$

Substituting the results of the previous step into it yields

$$20\theta_B + 90\theta_C + 41.7 = 0 \quad (b)$$

(4) Solve the primary unknowns.

Solving the two linear algebraic equations (a) and (b) simultaneously, we get

$$\theta_B = 1.15, \theta_C = -4.89$$

(5) Solve the member end moments.

Substituting the obtained member end displacements into the above member end moment equations, we have

$$\begin{aligned} M_{BA} &= 43.50 \text{ kN}\cdot\text{m} \\ M_{BE} &= 3.45 \text{ kN}\cdot\text{m}, M_{EB} = 1.73 \text{ kN}\cdot\text{m} \\ M_{BC} &= -46.90 \text{ kN}\cdot\text{m}, M_{CB} = 24.50 \text{ kN}\cdot\text{m} \\ M_{CD} &= -14.70 \text{ kN}\cdot\text{m} \\ M_{CF} &= -9.80 \text{ kN}\cdot\text{m}, M_{FC} = -4.89 \text{ kN}\cdot\text{m} \end{aligned}$$

It should be noted that because the relative stiffnesses of members are used, the displacements obtained in the example are not true values. To obtain the true values of the displacements, the stiffnesses should also be taken as true values.

6.4 Analysis of Frames with Sidesway

Frames can be divided into two types: without sidesway and with sidesway. This section discusses the internal force analysis of frames with sidesway. For the **frames with sidesway**, the primary unknowns usually have two types: the independent rotational displacement of rigid joint and the **linear displacement of the joint**. Therefore, compared with the calculation of frames without sidesway, the basic equations of displacement method are different for frames with sidesway. In addition to the basic equation corresponding to the joint rotational displacement (*i.e.*, the moment equilibrium equation), the basic equation corresponding to the static equilibrium condition by the joint linear displacement (*i.e.*, the sectional projection equilibrium equations) should also be established. The following is an example to illustrate how to establish the basic equations of displacement method for frames with sidesway.

Figure 6.14a demonstrates a frame, the linear stiffness of the column is i , and that of the beam is $2i$. Ignoring the axial deformation of the members, the primary unknowns of the displacement method are the angular displacement θ_B at joint B and the horizontal linear displacement Δ at the column top, as shown in figure 6.14b.

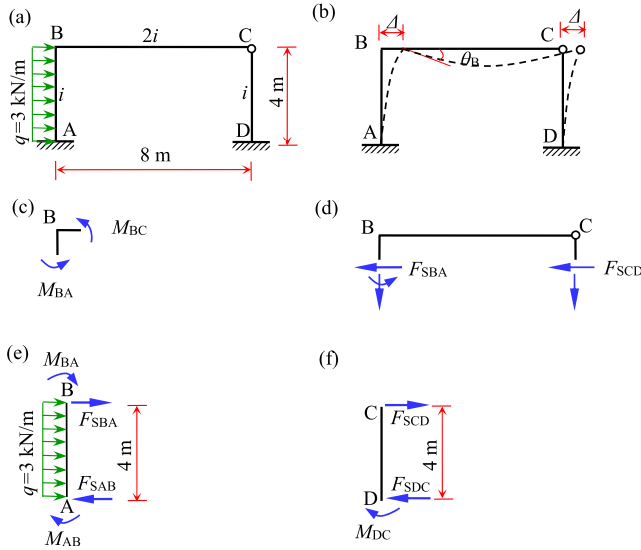


FIG. 6.14 – Analysis of frame with sidesway by displacement method.

When calculating the frame members, we should notice that there is relative translation Δ between the two ends of the members AB and CD. But both ends of member BC have only the overall horizontal displacement, without vertical displacement and relative transverse translation. By using the slope-deflection equations (6.8) and (6.9) and superimposing the fixed-end moments, the end moments of each member can be written as follows:

$$\left. \begin{aligned} M_{AB} &= 2i\theta_B - 6i\frac{\Delta}{4} - \frac{1}{12} \times 3 \times 4^2 \\ M_{BA} &= 4i\theta_B - 6i\frac{\Delta}{4} + \frac{1}{12} \times 3 \times 4^2 \\ M_{BC} &= 3(2i)\theta_B \\ M_{DC} &= -3i\frac{\Delta}{4} \end{aligned} \right\} \quad (a)$$

Now we can establish the basic equation of the displacement method. Firstly, for the angular displacement θ_B of joint B, taking joint B as a free body (figure 6.14c), the moment equilibrium equation can be written as

$$\sum M_B = 0, \quad M_{BA} + M_{BC} = 0 \quad (b)$$

Using equation (a), this moment equilibrium equation can be written as:

$$10i\theta_B - 1.5i\Delta + 4 = 0 \quad (6.14a)$$

Then, for the horizontal displacement Δ of the beam, taking the BC part of the beam above the column top as a free body (figure 6.14d), the horizontal projection equilibrium equation can be written as:

$$\sum F_x = 0, F_{SBA} + F_{SCD} = 0 \quad (c)$$

The member end shears in equation (c) can be transformed into the member end moments. To do this, take column AB as a free body (figure 6.14e, the member end axial forces are not drawn in the figure), we have:

$$\sum M_A = 0, F_{SBA} = -\frac{1}{4}(M_{AB} + M_{BA}) - 6$$

Taking column CD as a free body (figure 6.14f), we have

$$\sum M_D = 0, F_{SCD} = -\frac{1}{4}M_{DC}$$

Substituting the above two equations of shear forces into equation (c) yields

$$M_{AB} + M_{BA} + M_{DC} + 24 = 0 \quad (d)$$

By using equation (a), we have

$$6i\theta_B - 3.75i\Delta + 24 = 0 \quad (6.14b)$$

By solving the linear algebraic equations (6.14a) and (6.14b) simultaneously, the joint displacements θ_B and Δ can be obtained. Then, the member end moments can be obtained by substituting them into equation (a). Finally, we can draw the internal force diagram of the frame.

In general, the basic equations of the displacement method are derived from the equilibrium conditions. Among the primary unknowns of the displacement method, each angular displacement is related to a joint moment equilibrium equation, and each linear displacement is related to a section projection equilibrium equation. The number of equilibrium equations is equal to the number of primary unknowns, so all the primary unknowns can be solved.

Example 6.2. Draw the moment diagram of the bent structure shown in figure 6.15a, ignoring the axial deformation of the links.

Solution: (1) Determine the primary unknowns.

The columns AB, CD, and EF are parallel, so the links will only move horizontally after deformation. The links remain parallel before and after deformation (figure 6.15b), so the horizontal displacements at the top of each column are identical, *i.e.*, there is only one independent linear displacement Δ . There is no rigid joint in this example, so there is no angular displacement in the primary unknowns.

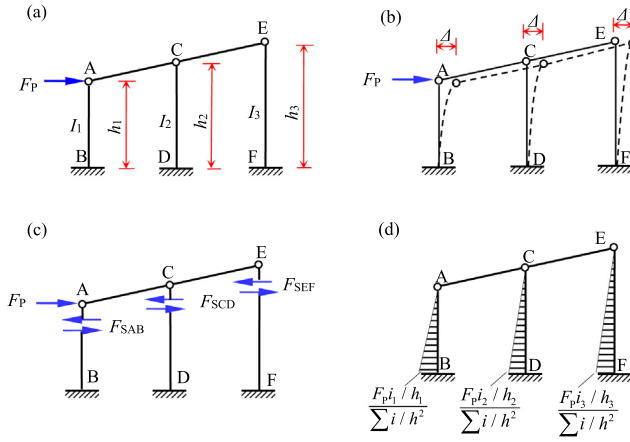


FIG. 6.15 – Analysis of the bent structure by displacement method.

(2) Calculate the end moments and the end shears of each member.

The linear stiffnesses of columns are: $i_1 = \frac{EI_1}{h_1}$, $i_2 = \frac{EI_2}{h_2}$, $i_3 = \frac{EI_3}{h_3}$.

From the slope-deflection equation (6.9), the member end moments are:

$$M_{BA} = -3i_1 \frac{\Delta}{h_1}, \quad M_{DC} = -3i_2 \frac{\Delta}{h_2}, \quad M_{FE} = -3i_3 \frac{\Delta}{h_3}$$

The end shears of each column can be obtained from its equilibrium condition:

$$F_{SAB} = 3i_1 \frac{\Delta}{h_1^2}, \quad F_{SCD} = 3i_2 \frac{\Delta}{h_2^2}, \quad F_{SEF} = 3i_3 \frac{\Delta}{h_3^2}$$

(3) Establish the displacement method equation.

Taking the links above the column top as a free body (figure 6.15c), according to the horizontal equilibrium condition $\sum F_x = 0$, we have

$$F_P - (F_{SAB} + F_{SCD} + F_{SEF}) = 0$$

$$F_P - 3\Delta \left(\frac{i_1}{h_1^2} + \frac{i_2}{h_2^2} + \frac{i_3}{h_3^2} \right) = 0$$

Then we have

$$\Delta = \frac{F_P}{3 \left(\frac{i_1}{h_1^2} + \frac{i_2}{h_2^2} + \frac{i_3}{h_3^2} \right)} = \frac{F_P}{3 \sum \frac{i}{h^2}}$$

where, $\sum \frac{i}{h^2}$ is the sum of each column's lateral stiffness $\frac{i}{h^2}$.

- (4) Solve the end moments and the end shears of each member.

Substituting Δ into the formulas in step (2) yields.

$$M_{BA} = -\frac{\frac{i_1}{h_1}}{\sum \frac{i}{h^2}} F_P, \quad M_{DC} = -\frac{\frac{i_2}{h_2}}{\sum \frac{i}{h^2}} F_P, \quad M_{FE} = -\frac{\frac{i_3}{h_3}}{\sum \frac{i}{h^2}} F_P$$

$$F_{SAB} = \frac{\frac{i_1}{h_1^2}}{\sum \frac{i}{h^2}} F_P, \quad F_{SCD} = \frac{\frac{i_2}{h_2^2}}{\sum \frac{i}{h^2}} F_P, \quad F_{SEF} = \frac{\frac{i_3}{h_3^2}}{\sum \frac{i}{h^2}} F_P$$

- (5) Draw the diagram M , as shown in figure 6.15d.
 (6) Discussion.

The calculation results show that the shear F_S of each column is proportional to the lateral stiffness $\frac{i}{h^2}$ when the bent is only loaded at the top of the columns. Therefore, the internal forces of the frame can be obtained by shear distribution method: the load F_P , as the total shears of each column, is distributed to each column according to the ratio of the lateral stiffness $\frac{i}{h^2}$ of each column. After the column top shears are obtained, the moment diagram can be drawn (figure 6.15d).

Example 6.3. Draw the internal force diagrams of the frame shown in figure 6.16a.

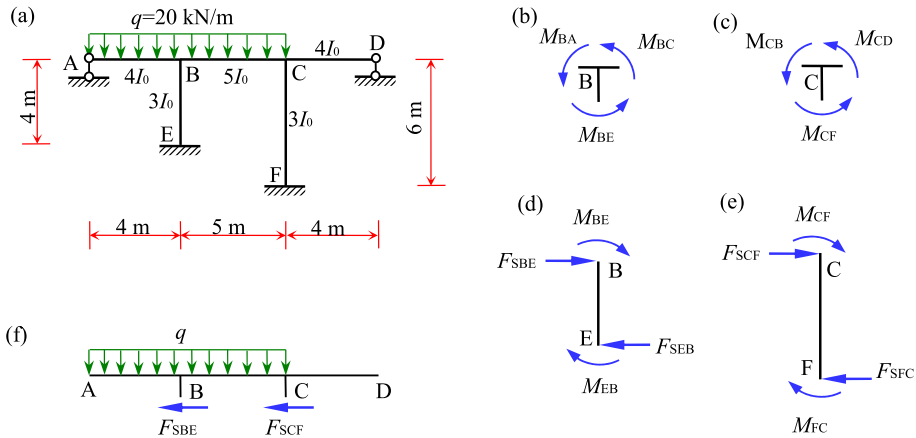


FIG. 6.16 – Analysis of the frame structure with sidesway.

Solution: (1) Determine the primary unknowns.

The difference between this example and example 6.1 is that the joints B and C have a horizontal linear displacement Δ in addition to the angular displacements θ_B and θ_C .

(2) Calculate the member end moments.

The fixed-end moments have been calculated in example 6.1. Let us still use the relative stiffness of each member. According to the slope-deflection equations (6.5), (6.8), and (6.9) and superimposing the fixed-end moments, the end moments of each member are:

$$M_{BA} = 3i_{BA}\theta_B + M_{BA}^F = 3\theta_B + 40.0$$

$$M_{BC} = 4i_{BC}\theta_B + 2i_{BC}\theta_C + M_{BC}^F = 4\theta_B + 2\theta_C - 41.7$$

$$M_{CB} = 2i_{BC}\theta_B + 4i_{BC}\theta_C + M_{CB}^F = 2\theta_B + 4\theta_C + 41.7$$

$$M_{CD} = 3i_{CD}\theta_C = 3\theta_C$$

$$M_{BE} = 4i_{BE}\theta_B - 6\frac{i_{BE}}{l_{BE}}\Delta = 3\theta_B - 1.125\Delta$$

$$M_{EB} = 2i_{BE}\theta_B - 6\frac{i_{BE}}{l_{BE}}\Delta = 1.5\theta_B - 1.125\Delta$$

$$M_{CF} = 4i_{CF}\theta_C - 6\frac{i_{CF}}{l_{CF}}\Delta = 2\theta_C - 0.5\Delta$$

$$M_{FC} = 2i_{CF}\theta_C - 6\frac{i_{CF}}{l_{CF}}\Delta = \theta_C - 0.5\Delta$$

(3) Establish the displacement method equations.

Considering the moment equilibrium of joint B (figure 6.16b),

$$\sum M_B = 0, \quad M_{BA} + M_{BC} + M_{BE} = 0$$

We have

$$10\theta_B + 2\theta_C - 1.125\Delta - 1.7 = 0 \quad (\text{a})$$

Considering the moment equilibrium of joint C (figure 6.16c),

$$\sum M_C = 0, \quad M_{CB} + M_{CD} + M_{CF} = 0$$

We have

$$2\theta_B + 9\theta_C - 0.5\Delta + 41.7 = 0 \quad (\text{b})$$

Cutting off the top of the columns with a section, and considering the equilibrium condition of the ABCD part of the beam above the top of the columns (figure 6.16f), we have:

$$\sum F_x = 0, F_{SBE} + F_{SCF} = 0$$

Considering the equilibrium of columns BE and CF (figure 6.16d, e), we obtain:

$$\sum M_E = 0, F_{SBE} = -\frac{M_{BE} + M_{EB}}{4}$$

$$\sum M_F = 0, F_{SCF} = -\frac{M_{CF} + M_{FC}}{6}$$

Therefore, the sectional projection equilibrium equation can be written as:

$$\frac{M_{BE} + M_{EB}}{4} + \frac{M_{CF} + M_{FC}}{6} = 0$$

That is

$$6.75\theta_B + 3\theta_C - 4.37\Delta = 0 \quad (c)$$

(4) Solve the primary unknowns.

Solving the three linear algebraic equations (a)–(c) simultaneously yields

$$\theta_B = 0.937, \theta_C = -4.946, \Delta = -1.946$$

(5) Solve the member end moments.

Substituting the obtained displacements into the equations in step (2), we get

$$\begin{aligned} M_{BA} &= 42.82 \text{ kN}\cdot\text{m} \\ M_{BE} &= 5.00 \text{ kN}\cdot\text{m}, M_{EB} = 3.59 \text{ kN}\cdot\text{m} \\ M_{BC} &= -47.82 \text{ kN}\cdot\text{m}, M_{CB} = 23.76 \text{ kN}\cdot\text{m} \\ M_{CD} &= -14.84 \text{ kN}\cdot\text{m} \\ M_{CF} &= -8.92 \text{ kN}\cdot\text{m}, M_{FC} = -3.97 \text{ kN}\cdot\text{m} \end{aligned}$$

(6) Draw the internal force diagrams.

Diagram M can be drawn from the member end moments, as shown in figure 6.17a. From the free body of each member, the end shears can be obtained by the equilibrium equations. And then the diagram F_S can be drawn (figure 6.17b). The end axial forces can be obtained from the equilibrium equations of the joints, and then the diagram F_N can be drawn (figure 6.17c).

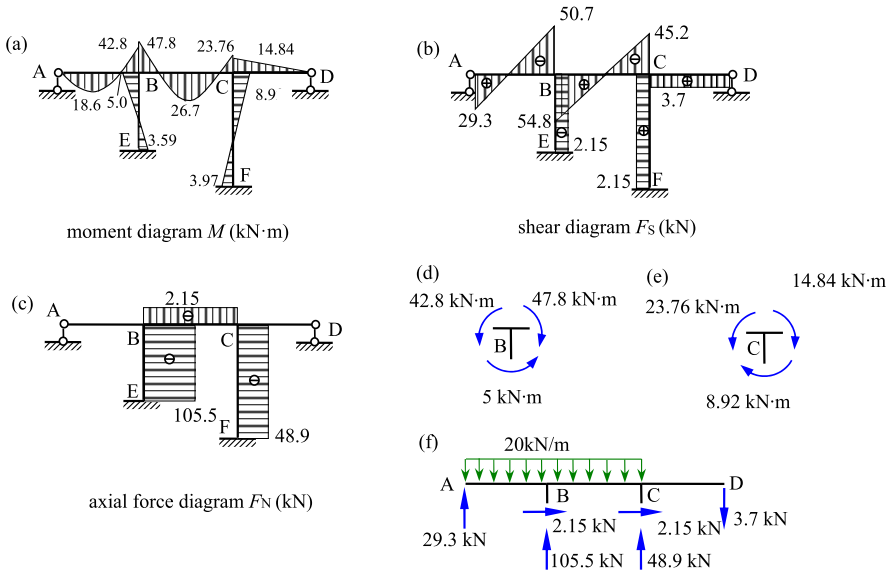


FIG. 6.17 – Internal force diagrams of frame and check of equilibrium conditions at joints B, C, and member AD.

(7) Check the results.

The check for calculated results of statically indeterminate structures by the force method has been discussed in detail before, and many of these methods are still applicable here. But we should note that, in the displacement method, the equilibrium conditions should be mainly checked; in contrast, in the force method, the continuous conditions of deformation are mainly checked. This is because the continuous conditions of deformation have been considered when we select the primary unknowns of the displacement method, and the calculation of the stiffness coefficients is relatively simple and not easy to make mistakes. Therefore, the continuous conditions of deformation are not the key point when checking the displacement method.

The internal force diagram in figure 6.17 can be checked by the equilibrium conditions. Firstly, it is seen in figure 6.17d and e that the moment equilibrium conditions at joints B and C are satisfied. Secondly, in figure 6.17f, by taking the ABCD part (that is the beam above the column top) as a free body, the horizontal and vertical equilibrium conditions can be checked:

$$\begin{aligned}\sum F_x &= 0, 2.15 \text{ kN} - 2.15 \text{ kN} = 0 \\ \sum F_y &= 0, 29.3 \text{ kN} + 105.5 \text{ kN} + 48.9 \text{ kN} - 20 \text{ kN/m} \\ &\quad \times 9 \text{ m} - 3.7 \text{ kN} = 183.7 \text{ kN} - 183.7 \text{ kN} = 0\end{aligned}$$

6.5 Analysis of Symmetric Structures

Symmetric continuous beams and frames are widely used in engineering. Any load acting on a symmetric structure can be divided into two parts: the symmetric load and the antisymmetric load. Under the action of symmetric load, the deformation is symmetric, the moment diagram and the axial force diagram are symmetric, but the shear diagram is antisymmetric. Under the antisymmetric load, the deformation is antisymmetric, the moment diagram and the axial force diagram are antisymmetric, but the shear diagram is symmetric. Section 5.5 has introduced how to select the half structure of a symmetric structure in the force method analysis. This section will introduce the use of symmetry in the displacement method to simplify the calculation of structures.

Example 6.4. Draw the moment diagram of the symmetric frame shown in figure 6.18a.

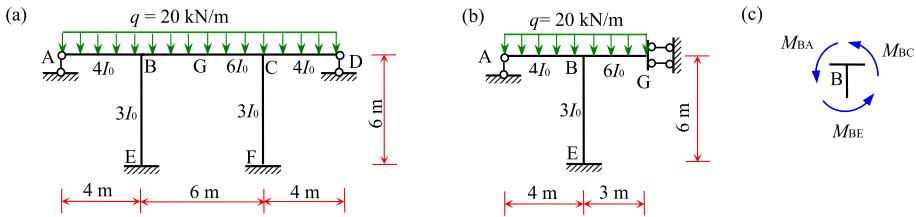


FIG. 6.18 – (a) A symmetric frame with odd spans, (b) half structure, and (c) equilibrium of joint B.

Solution: (1) Determine the primary unknowns.

This example is a symmetric structure with odd spans under symmetric loads. Due to the symmetry, the section G on the axis of symmetry has only symmetric internal forces and displacements (*i.e.*, the moment, axial force, and vertical displacement), and the antisymmetric internal forces and displacements (*i.e.*, the shears and horizontal displacements, rotation angles) are zero. Therefore, when constructing the half structure, a slider support should be added in section G. The half structure of the symmetric frame is shown in figure 6.18b. There is only one primary unknown in the displacement method, that is, the angular displacement θ_B of joint B.

(2) Calculate the member end moments.

The fixed-end moments can be gotten from table 6.2,

$$M_{BA}^F = \frac{ql^2}{8} = \frac{20 \text{ kN/m} \times (4 \text{ m})^2}{8} = 40.0 \text{ kN} \cdot \text{m}$$

$$M_{BG}^F = -\frac{ql^2}{3} = -\frac{20 \text{ kN/m} \times (3 \text{ m})^2}{3} = -60 \text{ kN} \cdot \text{m}$$

$$M_{GB}^F = -\frac{ql^2}{6} = -\frac{20 \text{ kN/m} \times (3 \text{ m})^2}{6} = -30 \text{ kN} \cdot \text{m}$$

Taking the relative stiffness of each member for calculation, and assuming $EI_0 = 1$, we have

$$i_{BA} = \frac{4EI_0}{4} = 1, \quad i_{BG} = \frac{6EI_0}{3} = 2, \quad i_{BE} = \frac{3EI_0}{6} = \frac{1}{2}$$

The end moments of each member are as follows:

$$M_{BA} = 3i_{BA}\theta_B + M_{BA}^F = 3\theta_B + 40.0$$

$$M_{BG} = i_{BG}\theta_B + M_{BG}^F = 2\theta_B - 60, \quad M_{GB} = -i_{BG}\theta_B + M_{GB}^F = -2\theta_B - 30$$

$$M_{BE} = 4i_{BE}\theta_B = 2\theta_B, \quad M_{EB} = 2i_{BE}\theta_B = \theta_B$$

(3) Establish the displacement method equation.

Considering the equilibrium of joint B (figure 6.18c),

$$\sum M_B = 0, \quad M_{BA} + M_{BC} + M_{BE} = 0$$

we have

$$7\theta_B - 20 = 0$$

(4) Solve the primary unknown.

$$\theta_B = \frac{20}{7} = 2.86$$

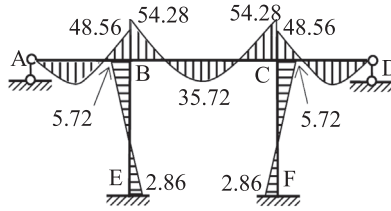
(5) Solve the member end moments.

Substituting the obtained angular displacement θ_B into the equations in step (2) yields

$$\begin{aligned} M_{BA} &= 48.56 \text{ kN} \cdot \text{m} \\ M_{BE} &= 5.72 \text{ kN} \cdot \text{m}, \quad M_{EB} = 2.86 \text{ kN} \cdot \text{m} \\ M_{BC} &= -54.28 \text{ kN} \cdot \text{m}, \quad M_{CB} = -35.72 \text{ kN} \cdot \text{m} \end{aligned}$$

(6) Draw the moment diagram.

The diagram M is shown in figure 6.19.

FIG. 6.19 – Moment diagram M (kN·m) of a symmetric frame.

Example 6.5. Draw the moment diagram of the symmetric frame shown in figure 6.20a. EI of each beam is identical.

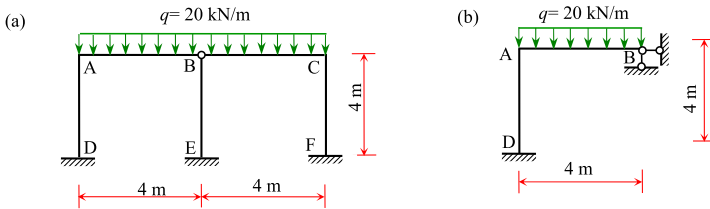


FIG. 6.20 – (a) Even-span symmetric frame and (b) half structure.

Solution:

- (1) Determine the primary unknowns.

This example is a symmetric structure with even spans under symmetric loads. Because of symmetry, the member BE at the position of the symmetry axis has only the axial force, and there is no moment at section B due to the pinned joint. Therefore, when constructing the half structure, fixed hinged support should be set in section B. The half structure of the symmetric frame is shown in figure 6.20b. There is only one primary unknown in the displacement method, that is, the angular displacement θ_A of joint A.

- (2) Calculate the member end moments.

The fixed-end moment can be gotten from table 6.2,

$$M_{AB}^F = -\frac{ql^2}{8} = -\frac{20 \text{ kN/m} \times (4 \text{ m})^2}{8} = -40.0 \text{ kN} \cdot \text{m}$$

The stiffness and length of each member are the same, so the linear stiffness i is the same. The end moments of each member are written as

$$M_{AB} = 3i\theta_A + M_{AB}^F = 3i\theta_A - 40.0, \quad M_{AD} = 4i\theta_A, \quad M_{DA} = 2i\theta_A$$

- (3) Establish the displacement method equation.

Considering the equilibrium of joint A,

$$\sum M_A = 0, \quad M_{AD} + M_{AB} = 0$$

we have

$$7i\theta_A - 40 = 0$$

- (4) Solve the primary unknown.

$$\theta_A = \frac{40}{7i}$$

- (5) Solve the member end moments.

Substituting the obtained angular displacement θ_A into the equations in step (2) yields

$$\begin{aligned} M_{AB} &= -22.86 \text{ kN}\cdot\text{m} \\ M_{AD} &= 22.86 \text{ kN}\cdot\text{m}, \quad M_{DA} = 11.43 \text{ kN}\cdot\text{m} \end{aligned}$$

- (6) Draw the moment diagram.

In terms of the member end moments, diagram M is exhibited in figure 6.21.

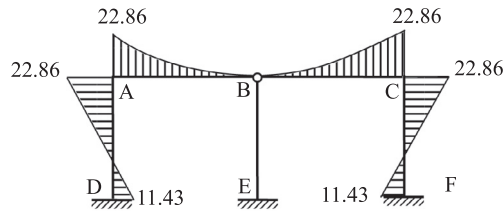


FIG. 6.21 – Moment diagram M (kN·m) of even-span symmetric frame.

6.6 Primary System in the Displacement Method

There are two ways to establish the basic equation of the displacement method: one is to write the equilibrium equation directly; the other is to establish the basic equation from the primary system of displacement method and equilibrium condition. The previous sections introduce in detail the fundamental principle and solution process of the first way of displacement method: firstly, establish an equilibrium

equation, which is expressed by the member end internal forces; then, the member end internal forces are expressed by the joint displacements, thus the equilibrium equation expressed by joint displacements can be obtained. This is the basic equation of the displacement method. There is no application of the concept of a primary system.

This section introduces the second way of displacement method, including the primary system and the basic equation of displacement method: firstly, establish the primary system of displacement method, and then establish the basic equations of displacement method (also called the canonical equations) through the primary system. This method is similar to the establishment of the basic equation in the force method, which helps to deeply understand the meaning of the basic equation of the displacement method and make some preparations in advance for the matrix displacement method that will be introduced later.

In what follows, combined with the frame shown in figure 6.22a, the two points are highlighted: (1) how to establish the primary system of displacement method; (2) how to construct the basic equation of displacement method. This frame has been analyzed in section 6.4, as shown in figure 6.14. We can study it by comparison.

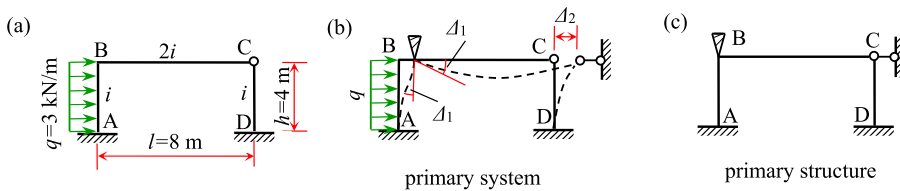


FIG. 6.22 – (a) A frame structure, (b) primary system, and (c) primary structure.

The frame has two primary unknowns: the rotational angle Δ_1 of joint B and the horizontal displacement Δ_2 of joint C. Here, the primary unknowns of the displacement method, regardless of angular displacements or linear displacements, are uniformly represented by Δ , so as to contrast with the primary unknown X in the force method.

6.6.1 Primary System of the Displacement Method

Figure 6.22b is the primary system of the displacement method: a rigid arm is added at rigid joint B to control its rotational angle (the linear displacement is not controlled), and a horizontal strut is added at joint C to control its horizontal displacement.

The difference between the **primary system of displacement method** and the original structure is that: artificial constraints corresponding to the primary unknowns are added to make the primary unknowns change from the passive displacements to the artificially controlled active displacements.

In the primary system of displacement method, if we remove the load and just look at the structure, we can get the structure shown in figure 6.22c, which is called the primary structure. The **primary structure of the displacement method** is a structure obtained by adding some controllable restraints corresponding to the primary unknowns on the original structure.

The primary system is a tool or bridge for calculating the original structure. On the one hand, it can be transformed into the original structure, which means it can represent the original structure; on the other hand, its calculation is relatively simple, so as to “make hard things simple”. Due to the increase of artificial constraints, the original entire structure is divided into many members (these members deform independently and do not interfere with each other, and their slope-deflection equations have been derived in advance). Accordingly, the calculation of the structure is divided into the calculation of many members, which simplifies the calculation. Therefore, the primary system of the displacement method is a set of members with additional restraints, which is identical to the original structure in terms of load and deformation characteristics.

It should be noted that in the force method, we remove restraints to simplify the calculation of structural internal forces. In the displacement method, however, we add restraints to achieve the purpose of simplifying the calculation of internal forces. The measures are the opposite, but the effects are the same.

6.6.2 Basic Equations of the Displacement Method

The basic equations of the displacement method can be established from the primary system of the displacement method.

Under what conditions, can the primary system be transformed to the original structure? This transformation condition is just the basic equation of the displacement method. Now we consider this problem in two steps.

The first step is to control the additional restraints so that the joint displacements Δ_1 and Δ_2 are zero. Under this condition, the primary structure is in a locked state. After the load is applied, the internal forces in the primary structure can be solved (figure 6.23a). At the same time, additional restraints will produce a restraint moment F_{1P} and a restraint horizontal force F_{2P} . These restraint forces do not exist in the original structure.

The second step is to control the additional restraints so that the primary structure has joint displacements Δ_1 and Δ_2 . In this case, the restraint forces F_1 and F_2 in the additional restraints will change accordingly. If the joint displacements Δ_1 and Δ_2 are controlled to be exactly identical to the actual values of the original structure, the constraint forces F_1 and F_2 will disappear completely. Then the primary system shown in figure 6.22b can be obtained. Though the primary system still has additional restraints, they are actually indistinguishable. Hence, the primary system is actually in a relaxed state, which is exactly the same as the original structure.

Therefore, the conditions for transforming the primary system into the original structure are: under the combined action of a given load and joint displacements Δ_1

and Δ_2 , for the primary structure, the total restraint forces F_1 and F_2 generated in the additional restraints should be equal to zero, that is

$$\left. \begin{array}{l} F_1 = 0 \\ F_2 = 0 \end{array} \right\} \quad (6.15)$$

Clearly, this transformation condition leads to the basic equations of the displacement method.

Based on the superposition principle, the calculation of total constraint forces in the primary system can be divided into several cases:

- (1) The loads are applied alone—the corresponding restraint forces are F_{1P} and F_{2P} (figure 6.23a).
- (2) Unit displacement $\Delta_1 = 1$ has been imposed alone—the corresponding restraint forces are k_{11} and k_{21} (figure 6.23b).
- (3) Unit displacement $\Delta_2 = 1$ is imposed alone—the corresponding restraint forces are k_{12} and k_{22} (figure 6.23c).

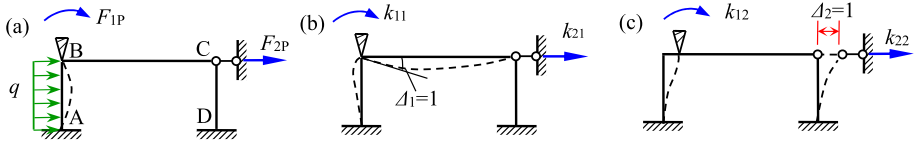


FIG. 6.23 – Primary structure under (a) loads, (b) unit displacement $\Delta_1 = 1$, (c) unit displacement $\Delta_2 = 1$.

The total restraint forces can be obtained by superimposing the above results

$$\left. \begin{array}{l} F_1 = k_{11}\Delta_1 + k_{12}\Delta_2 + F_{1P} \\ F_2 = k_{21}\Delta_1 + k_{22}\Delta_2 + F_{2P} \end{array} \right\} \quad (6.16)$$

Considering equation (6.15), the basic equations of the displacement method are written as:

$$\left. \begin{array}{l} k_{11}\Delta_1 + k_{12}\Delta_2 + F_{1P} = 0 \\ k_{21}\Delta_1 + k_{22}\Delta_2 + F_{2P} = 0 \end{array} \right\} \quad (6.17)$$

The primary unknowns Δ_1 and Δ_2 can be solved from the basic equations.

From the perspective of the primary system, the basic equations have a clear meaning, that is, the additional restraints in the primary system should actually be in a relaxed state, and the restraint forces in the additional restraints should all be zero; in fact, it requires the original structure to satisfy the equilibrium equation.

In fact, the basic idea of the displacement method is still a transition method, that is, the transition from the primary system to the original structure. The transition step is to first lock and then relax, and then establish the basic equations

of the displacement method according to the relaxed conditions. The idea of “first lock and then relax” mentioned here and the “first disassemble and then assemble” mentioned earlier summarize the basic ideas of the displacement method from different perspectives; but the two ideas are interrelated. “Lock” is actually “disassemble” the overall deformation of the structure into isolated member deformation. “Relax” is to require the additional restraints ineffective, that is to require the equilibrium condition to be satisfied when the members are assembled together.

6.6.3 Process of Establishing the Basic Equations of the Displacement Method

Now we will follow the above steps to apply the displacement method for structural analysis and calculation.

(1) Calculation of the primary structure under loads.

Firstly, the fixed-end moments of each member are calculated respectively, and the moment diagram is drawn in figure 6.24a. The moment diagram of the primary structure under loads is called the diagram M_P .

Taking joint B as a free body (figure 6.24b), it can be obtained that $F_{1P} = 4 \text{ kN}\cdot\text{m}$.

Taking beam BC as a free body (figure 6.24c), as the fixed-end shear of column BA $F_{SBA} = -\frac{qh}{2} = -\frac{3 \times 4}{2} = 6 \text{ kN}$ is known, we have $F_{2P} = -6 \text{ kN}$.

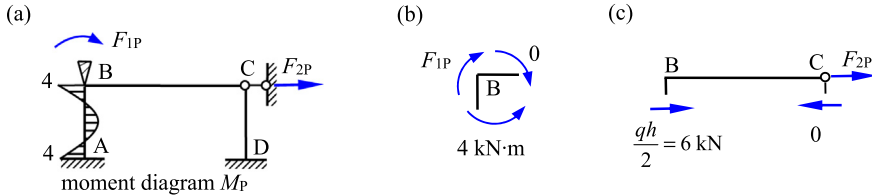


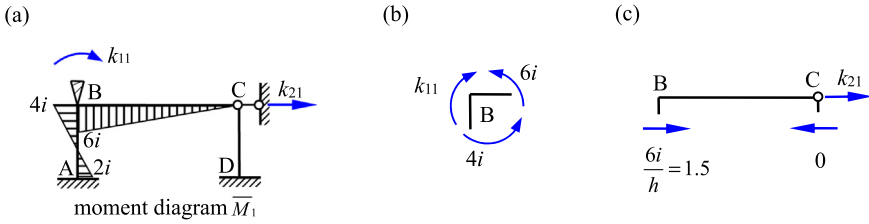
FIG. 6.24 – Primary structure under loads.

(2) Calculation of the primary structure under unit angular displacement $\Delta_1 = 1$.

When the angular displacement of joint B is $\Delta_1 = 1$, the end moments of each member are calculated and the moment diagram (diagram $\bar{M}_1$) can be drawn in figure 6.25a.

From figure 6.25b and c, we can get

$$k_{11} = 4i + 3(2i) = 10i, k_{21} = -1.5i$$

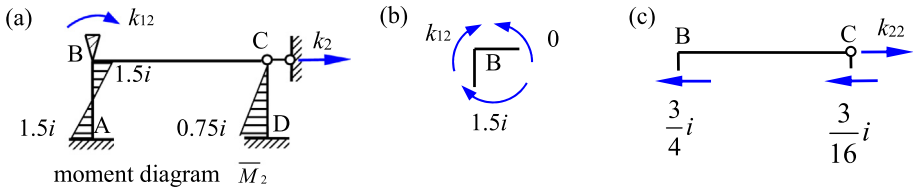
FIG. 6.25 – Primary structure under unit horizontal displacement $\Delta_1 = 1$.

- (3) Calculation of the primary structure under unit horizontal displacement $\Delta_2 = 1$.

When the horizontal displacement of joints B and C is $\Delta_2 = 1$, the end moments of each member are calculated and the moment diagram (diagram $\bar{M}_2$) can be drawn in figure 6.26a.

From figure 6.26b and c, we can get

$$k_{12} = -1.5i, \quad k_{22} = \frac{15}{16}i$$

FIG. 6.26 – Primary structure under unit horizontal displacement $\Delta_2 = 1$.

- (4) Establishment of the basic equations of the displacement method.

According to equation (6.17), the basic equations can be written as follows:

$$10i\Delta_1 - 1.5i\Delta_2 + 4 = 0$$

$$-1.5i\Delta_1 + \frac{15}{16}i\Delta_2 - 6 = 0$$

The basic equations obtained here are the same as equations (6.14a) and (6.14b) in section 6.4.

Solving the basic equations simultaneously yields:

$$\Delta_1 = 0.737 \frac{1}{i}, \quad \Delta_2 = 7.580 \frac{1}{i}$$

By using the following superposition formula, the total moment of the frame is:

$$M = \overline{M}_1 A_1 + \overline{M}_2 A_2 + M_P \quad (6.18)$$

The member end moments are:

$$M_{AB} = 2i \times \left(0.737 \frac{1}{i}\right) - 1.5i \times \left(7.580 \frac{1}{i}\right) - 4 = -13.90 \text{ kN}\cdot\text{m}$$

$$M_{BA} = 4i \times \left(0.737 \frac{1}{i}\right) - 1.5i \times \left(7.580 \frac{1}{i}\right) + 4 = -4.42 \text{ kN}\cdot\text{m}$$

$$M_{BC} = 6i \times \left(0.737 \frac{1}{i}\right) = 4.42 \text{ kN}\cdot\text{m}$$

$$M_{DC} = -0.75i \times \left(7.580 \frac{1}{i}\right) = -5.69 \text{ kN}\cdot\text{m}$$

According to the member end moments, the diagram M of the frame can be drawn, as shown in figure 6.27.

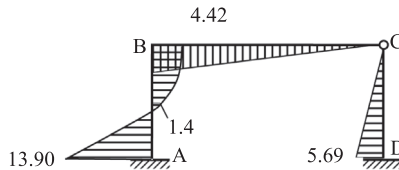


FIG. 6.27 – Moment diagram M (kN·m) of frame.

6.6.4 Canonical Equations of Displacement Method

In the above sections, the frame with two primary unknowns is taken as an example, and the meanings of the primary system and basic equations in the displacement method are illustrated. For a problem with n primary unknowns, the basic equations of the displacement method can be written as follows in accordance with equation (6.17):

$$\left. \begin{aligned} k_{11}A_1 + k_{12}A_2 + \cdots + k_{1n}A_n + F_{1P} &= 0 \\ k_{21}A_1 + k_{22}A_2 + \cdots + k_{2n}A_n + F_{2P} &= 0 \\ &\dots\dots\dots \\ k_{n1}A_1 + k_{n2}A_2 + \cdots + k_{nn}A_n + F_{nP} &= 0 \end{aligned} \right\} \quad (6.19)$$

Equation (6.19) is written according to certain rules. It does not depend on the form of the structure and has the same form as the canonical equations of force method. Therefore, it is called the **canonical equations of displacement method**.

The canonical equation (6.19) of the displacement method can be written in the matrix form:

$$\mathbf{K}\mathbf{A} + \mathbf{F}_P = \mathbf{0} \quad (6.20)$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$$

is called the **stiffness matrix of the structure**, and the corresponding coefficients are named after the stiffness coefficients. From the theorem of reciprocal reactions, we know:

$$k_{ij} = k_{ji}$$

Therefore, the structural stiffness matrix is also a symmetric matrix, and the coefficients k_{ii} on the diagonal are called the primary coefficients. They represent the reaction forces (or reaction moments) of the additional restraint i caused by the unit displacement $\Delta_i = 1$ at the position of the additional restraint i of the primary structure. The direction of the reaction force (or reaction moment) is consistent with the direction of the assumed Δ_i , so the primary coefficients are always positive. The other coefficients k_{ij} are called the secondary coefficients. They represent the reaction force (or reaction moment) on the additional constraint i caused by the unit displacement $\Delta_j = 1$ at the position of the additional constraint j . Their signs depend on whether it is in the same direction as the assumed Δ_i , so they can be positive, negative, or zero. Moreover, the quadratic form derived from the structural stiffness matrix and the non-zero displacement vector represents the strain energy of the structure, which is always positive, hence $\mathbf{K}$ is also a positive definite matrix. The canonical equations of the displacement method are also named the stiffness equations of the structure, which is essentially a set of equilibrium equations. It is worth pointing out that the canonical equations of the displacement method and the canonical equations of the force method reflect the beauty of symmetry, simplicity, and unity in mechanics.

$\mathbf{A}$ is the primary unknown displacement vector, and $\mathbf{F}_P$ is the constraint force vector generated by the additional constraint. The free term F_{iP} represents the reaction force (or reaction moment) generated on the additional constraint i of the primary structure when the load acts alone, so it is also called the loaded term. Its sign also depends on whether it is consistent with the direction of the assumed Δ_i . If it is true, the sign is positive, otherwise negative.

The displacement method equations are a set of linear algebraic equations. All the primary unknowns can be obtained by solving the equations, that is, the

independent joint displacements. Then, the member end moments and member end shears can generally be calculated directly by the slope-deflection equations, or they can be calculated by equation (6.21) according to the superposition principle:

$$\left. \begin{aligned} M &= \overline{M}_1 \Delta_1 + \overline{M}_2 \Delta_2 + \cdots + \overline{M}_n \Delta_n + M_P \\ F_S &= \overline{F}_{S1} \Delta_1 + \overline{F}_{S2} \Delta_2 + \cdots + \overline{F}_{Sn} \Delta_n + F_{SP} \\ F_N &= \overline{F}_{N1} \Delta_1 + \overline{F}_{N2} \Delta_2 + \cdots + \overline{F}_{Nn} \Delta_n + F_{NP} \end{aligned} \right\} \quad (6.21)$$

where $\overline{M}_i$, $\overline{F}_{Si}$ and $\overline{F}_{Ni}$ in the equations represent the internal forces generated by $\Delta_i = 1$ in the primary structure. M_P , F_{SP} and F_{NP} are the internal forces generated by the loads in the primary structure.

6.7 Principle of Potential Energy and Displacement Method

In mechanical analysis, the displacement method is the core. Problems of structural mechanics, elastic mechanics, and fluid mechanics can all be solved by the displacement method. The energy principle is the theoretical basis of mechanical analysis. The principle of potential energy is very important in structural analysis, and it is the basic energy principle corresponding to the displacement method. Applying the principle of potential energy not only can derive the equilibrium equation of the displacement method, so as to calculate the internal forces of structures, but also analyze and solve problems such as buckling stability of structures. This solution is another important method of structural mechanics: the energy method, which is the key means to solving complex mechanical problems. The energy method is aimed at the whole structural system. Considering the energy relationship of the system, the equilibrium equations and geometric equations are replaced with the corresponding virtual work equations or energy equations, and some functional variation equations are established. Thus, the structural mechanic's problem is reduced to the variation problem of obtaining the functional extreme (stationary) value under the given constraints. This section focuses on applying the principle of potential energy to establish the equilibrium equation of the displacement method.

6.7.1 Principle of Stationary Potential Energy

(1) Total potential energy of elastic structure system

The total potential energy of an elastic structure system is equal to the work done by all forces when the system recovers unloaded from the loaded state (*i.e.* the original state). The forces of the system include external forces and internal forces. If the total potential energy of an elastic system is denoted by Π , then the total potential energy of a general elastic structure system is

$$\Pi = U - \sum_{i=1}^n P_i \Delta_i \quad (\text{a})$$

where, U is the **strain energy** stored in the system during the loading deformation process, which is equal to the work done by the internal forces of the system in the process of unloading; $-\sum_{i=1}^n P_i \Delta_i$ is the work done by the external forces when the system recovers from the final deformed state to the unloaded state, and it is usually called the **potential energy of external force**. The minus sign indicates that the direction of external forces is opposite to the direction of deformation and displacement during unloading.

(2) Principle of stationary potential energy of elastic system

Assuming that the elastic structure system is in equilibrium, if any small displacement that satisfies geometric conditions (including deformation continuity conditions and displacement boundary conditions) occurs in the system, the variation of total potential energy is expressed as

$$\begin{aligned} \delta\Pi &= \delta\left(U - \sum_{i=1}^n P_i \Delta_i\right) \\ &= \delta U - \sum_{i=1}^n P_i \delta\Delta_i \\ &= \sum \int (M\delta\theta + F_N\delta u + F_S\delta\gamma) - \sum_{i=1}^n P_i \delta\Delta_i \end{aligned} \quad (\text{b})$$

where, δ is the variational symbol.

Since the above displacements $\delta\theta$, δu , $\delta\gamma$, $\delta\Delta$ are arbitrary small displacements that conform to the structural constraints, and the system satisfies the equilibrium conditions of the force system, the virtual work equation of the deformed body can be established:

$$\sum \int (M\delta\theta + F_N\delta u + F_S\delta\gamma) = \sum_{i=1}^n P_i \delta\Delta_i \quad (\text{c})$$

Comparing with equations (b) and (c), we can get

$$\delta\Pi = \delta\left(U - \sum_{i=1}^n P_i \Delta_i\right) = 0 \quad (6.22)$$

This formula is called the stationary condition of the potential energy of the elastic structure system. It shows that when an elastic structure system is in equilibrium, the first-order variation of any total potential energy that satisfies the geometric conditions must be zero. In other words, for the total potential energy in an equilibrium structural system, the necessary condition for the existence of a stationary value is that its first-order variation is zero.

The **principle of stationary potential energy** of an elastic structural system, simply referred to as the **principle of potential energy**, is expressed as follows: in all

possible displacements that satisfy the geometric conditions, the true displacement makes the potential energy be a stationary value; in return, the possible displacement that makes the potential energy a stationary value is the true displacement.

According to the classical criterion of equilibrium stability, when the second-order variation of total potential energy $\delta^2\Pi > 0$, it indicates that the total potential energy is the smallest, and the system is in a stable equilibrium state; When $\delta^2\Pi = 0$, the system is in any equilibrium state; When $\delta^2\Pi < 0$, it indicates that the total potential energy is the largest, and the system is in an unstable equilibrium state.

In the process of discussing the principle of stationary potential energy, the principle of potential energy is applicable to linear elastic system, nonlinear elastic system and elastoplastic deformation system because the properties of material are not involved. For a linear elastic structure system, the true displacement not only makes the potential energy a stationary value, but also makes the potential energy a minimum value. Therefore, not only the principle of stationary potential energy is established, but also the principle of minimum potential energy is also established.

6.7.2 Linear Elastic Strain Energy of Prismatic Member

(1) Truss member

The linear elastic strain energy of a truss member is

$$U = \frac{1}{2} \frac{F_N^2 l}{EA}$$

Substituting $F_N = \frac{EAu}{l}$ into the above equation, we can get

$$U = \frac{1}{2} \frac{EA}{l} u^2 \quad (6.23)$$

where, u is the axial displacement caused by F_N .

(2) Bending member with both fixed ends

When there is no load on a prismatic member fixed at both ends, and there are only the end rotations θ_A , θ_B and the lateral relative displacement Δ_{AB} , the member end forces can be calculated by equations (6.5) and (6.6)

$$M_{AB} = 4i\theta_A + 2i\theta_B - 6i\frac{\Delta_{AB}}{l}$$

$$M_{BA} = 2i\theta_A + 4i\theta_B - 6i\frac{\Delta_{AB}}{l}$$

$$F_{SBA} = F_{SAB} = -6i\frac{\theta_A}{l} - 6i\frac{\theta_B}{l} + 12i\frac{\Delta_{AB}}{l^2}$$

Then the strain energy can be calculated

$$\begin{aligned}
 U &= \frac{1}{2} M_{AB}\theta_A + \frac{1}{2} M_{BA}\theta_B + \frac{1}{2} F_{SBA}\Delta_{AB} \\
 &= 2i(\theta_A^2 + \theta_B^2 + \theta_A\theta_B) - \frac{6i}{l}\left(\theta_A + \theta_B - \frac{\Delta_{AB}}{l}\right)\Delta_{AB}
 \end{aligned} \tag{6.24}$$

(3) Bending members with one fixed end and another pinned end

When there is no load on a prismatic member with one end fixed and the other end pinned, only the end rotation θ_A and lateral relative displacement Δ_{AB} exist, the resulting end forces can be calculated by equations (6.9) and (6.6)

$$\begin{aligned}
 M_{AB} &= 3i\theta_A - 3i\frac{\Delta_{AB}}{l} \\
 F_{SBA} &= -3i\frac{\theta_A}{l} + 3i\frac{\Delta_{AB}}{l^2}
 \end{aligned}$$

Then the strain energy is

$$\begin{aligned}
 U &= \frac{1}{2} M_{AB}\theta_A + \frac{1}{2} F_{SBA}\Delta_{AB} \\
 &= \frac{3i}{2}\left(\theta_A^2 - \frac{2}{l}\theta_A\Delta_{AB} + \frac{1}{l^2}\Delta_{AB}^2\right)
 \end{aligned} \tag{6.25}$$

(4) Bending members with one fixed end and another slider-supported end

For a bending prismatic member with one end fixed and the other end slider-supported, if only the end rotation θ_A exists, the resulting end moment can be calculated by equation (6.10)

$$M_{AB} = i\theta_A$$

Then the strain energy is

$$U = \frac{1}{2} M_{AB}\theta_A = \frac{1}{2} i\theta_A^2 \tag{6.26}$$

6.7.3 Principle of Potential Energy and Equilibrium Equation of Displacement Method

From the process of demonstrating the principle of stationary potential energy, we know that the principle of stationary potential energy is essentially another expression of the principle of virtual displacement of deformable structures. Since the principle of virtual displacement is equivalent to the equilibrium condition,

the principle of potential energy is also equivalent to the equilibrium condition. The principle of potential energy is equivalent to the force equilibrium equation expressed by the functional stationary condition. If the potential energy is expressed as a function of the joint displacements in an elastic structural system, the equilibrium equation of the system expressed by the joint displacements can be established according to the principle of potential energy. Therefore, the principle of potential energy is similar to the displacement method. An example is taken to illustrate the use of the principle of potential energy to derive the equilibrium equation of the displacement method and to calculate the internal forces and displacements.

Example 6.6. Figure 6.28a shows a truss structure and tensile stiffness EA of each member is identical. Determine the internal forces of members by the principle of potential energy.

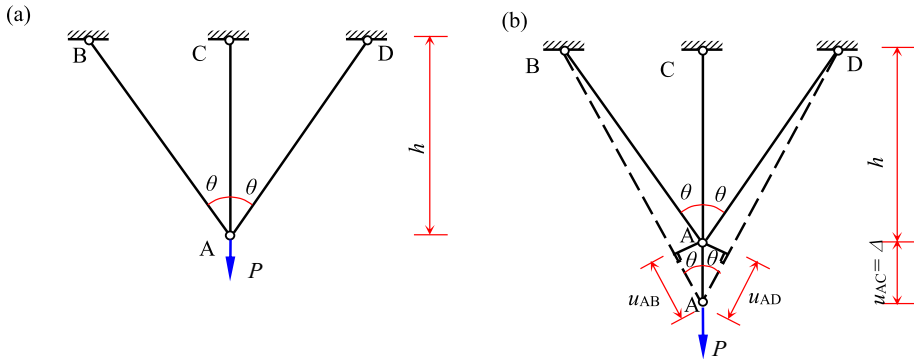


FIG. 6.28 – Truss structure with the same tensile stiffness EA for each member.

Solution:

Denote the elongation of the members as u , assume the displacement of joint A under the vertical load P is Δ , and assume the change of θ is small, which can be omitted. From the geometric relationship of figure 6.28b, we can get

$$u_{AC} = \Delta, \quad u_{AB} = u_{AD} = \Delta \cos \theta$$

Then, according to equation (6.23), the linear elastic strain energy of the system is

$$\begin{aligned} U &= \frac{1}{2} \sum \frac{EA}{h} u^2 \\ &= \frac{1}{2} \frac{EA}{h} \Delta^2 + \frac{1}{2} \frac{EA \cos \theta}{h} (\Delta \cos \theta)^2 \times 2 \\ &= \frac{EA}{2h} (1 + 2 \cos^3 \theta) \Delta^2 \end{aligned}$$

The total potential energy of the system is

$$\begin{aligned}\Pi &= U - \sum_{i=1}^n P_i \Delta_i \\ &= \frac{EA}{2h} (1 + 2 \cos^3 \theta) \Delta^2 - P \Delta\end{aligned}$$

This formula shows that the total potential energy is a quadratic function of the displacement Δ , therefore the variation operation of the potential energy can be transformed into a differential operation, so we get

$$\frac{d\Pi}{d\Delta} = \frac{EA}{h} (1 + 2 \cos^3 \theta) \Delta - P = 0$$

This formula is the equilibrium equation of the displacement method. By solving it we get

$$\Delta = \frac{Ph}{EA(1 + 2 \cos^3 \theta)}$$

Thereby, the axial force of truss members can be obtained:

$$F_{NAC} = \frac{EA}{h} \Delta = \frac{P}{1 + 2 \cos^3 \theta}$$

$$F_{NAB} = F_{NAD} = \frac{EA \cos \theta}{h} \Delta \cos \theta = \frac{P \cos^3 \theta}{1 + 2 \cos^3 \theta}$$

Example 6.7. Use the principle of potential energy to solve the joint displacements of the frame shown in figure 6.29a.

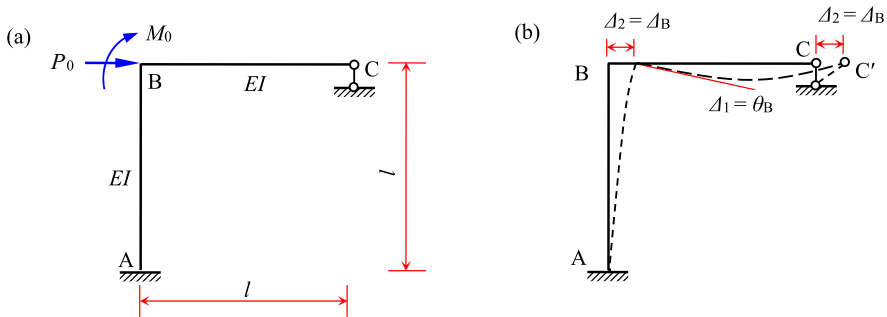


FIG. 6.29 – (a) Frame structure, (b) two displacement unknowns.

Solution: Joint B has two unknowns: linear displacement Δ_B and rotational displacement θ_B . In order to make a comparison with the displacement method, take $\Delta_1 = \theta_B$ and $\Delta_2 = \Delta_B$. From equations (6.24) and (6.25), the elastic strain energy of the system can be written as

$$\begin{aligned}
 U &= U_{AB} + U_{BC} \\
 &= \left[2iA_1^2 - \frac{6i}{l} \left(A_1 - \frac{A_2}{l} \right) A_2 \right] + \frac{3i}{2} A_1^2 \\
 &= \frac{7}{2} i A_1^2 - \frac{6i}{l} A_1 A_2 + \frac{6i}{l^2} A_2^2
 \end{aligned}$$

The external potential energy is

$$- \sum P\Delta = -M_0 A_1 - P_0 A_2$$

The total potential energy is

$$\Pi = \frac{7}{2} i A_1^2 - \frac{6i}{l} A_1 A_2 + \frac{6i}{l^2} A_2^2 - M_0 A_1 - P_0 A_2$$

This formula shows the total potential energy is a quadratic function of the joint displacements A_1 and A_2 , hence the variation operation of potential energy can be transformed to the partial differential operation.

$$\frac{\partial \Pi}{\partial A_1} = 7iA_1 - \frac{6i}{l} A_2 - M_0 = 0$$

$$\frac{\partial \Pi}{\partial A_2} = -\frac{6i}{l} A_1 + \frac{12i}{l^2} A_2 - P_0 = 0$$

The above equations are the equilibrium equations of displacement method, in which the primary coefficients $k_{11} = 7i$, $k_{22} = \frac{12i}{l^2}$, secondary coefficients $k_{12} = k_{21} = -\frac{6i}{l}$, free terms $P_1 = -M_0$, $P_2 = -P_0$. Solving the equations simultaneously yields

$$\begin{aligned}
 A_1 &= \frac{l^2}{8EI} \left(P_0 + \frac{2M_0}{l} \right) \\
 A_2 &= \frac{l^3}{12EI} \left(\frac{7P_0}{4} + \frac{3M_0}{2l} \right)
 \end{aligned}$$

Then, the internal force of each member can be easily solved.

Now let us observe the second-order variation $\delta^2 \Pi$ of the total potential energy. Since Π is a quadratic function of displacements, we can transform the second-order variation operation to the second-order partial differential operation, that is

$$\frac{\partial^2 \Pi}{\partial A_1^2} = 7i = k_{11} > 0$$

$$\frac{\partial^2 \Pi}{\partial A_2^2} = \frac{12i}{l^2} = k_{22} > 0$$

or generally written as

$$\frac{\partial^2 \Pi}{\partial \Delta_i^2} = k_{ii} > 0$$

This formula shows that the second-order variation of the total potential energy of the elastic system $\delta^2 \Pi > 0$, indicating that the total potential energy takes a minimum value. So the principle of potential energy is just the principle of minimum potential energy. The total potential energy of the system takes a minimum value, which indicates that the system is in a stable equilibrium state.

Example 6.8. Utilize the principle of potential energy to solve the joint displacements θ_B and θ_C of the continuous beam shown in figure 6.30a.

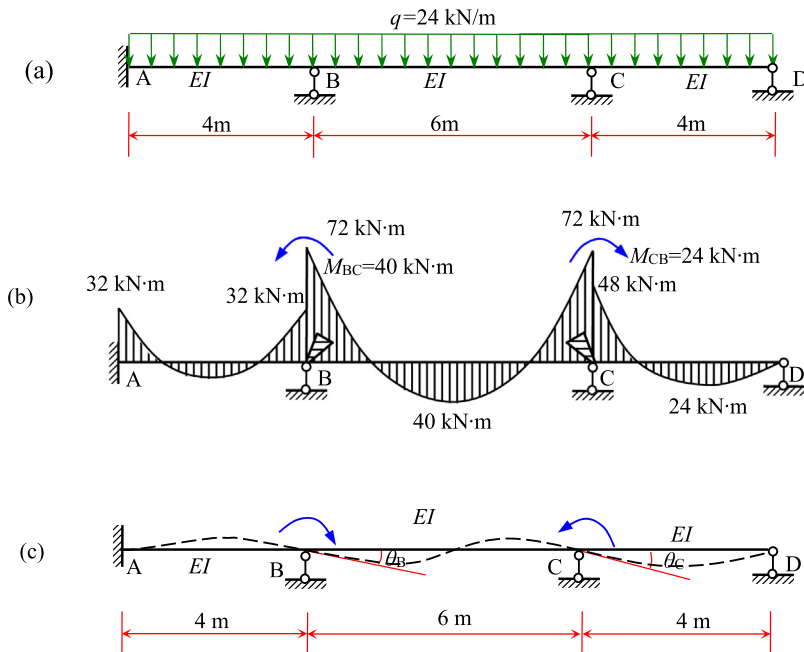


FIG. 6.30 – (a) A continuous beam, (b) rigid arms and reaction moments, and (c) computational model under equivalent joint loads.

Solution: This structure is subjected to non-joint loads. To use the principle of potential energy to establish the equation of displacement method, rigid arms must be placed on joints B and C (figure 6.30b), and the reaction moments in the additional rigid arms should be determined: $P_1 = 40 \text{ kN}\cdot\text{m}$ (↺) and $P_2 = 24 \text{ kN}\cdot\text{m}$ (↻). Then the reaction moments of the additional rigid arms should be applied at the joints with an opposite direction, so the computational model under equivalent joint loads can be obtained, as shown in figure 6.30c. Assuming that the direction of

angular displacements θ_B and θ_C at joints B and C is clockwise, the strain energy of the system can be obtained from equations (6.24) and (6.25) as

$$\begin{aligned} U &= U_{AB} + U_{BC} + U_{CD} \\ &= 2 \times \frac{EI}{4} \theta_B^2 + 2 \times \frac{EI}{6} (\theta_B^2 + \theta_B \theta_C + \theta_C^2) + \frac{3}{2} \times \frac{EI}{4} \theta_C^2 \\ &= \frac{5EI}{6} \theta_B^2 + \frac{EI}{3} \theta_B \theta_C + \frac{17EI}{24} \theta_C^2 \end{aligned}$$

Accordingly, the total potential energy of the system is

$$\begin{aligned} \Pi &= U - \sum P\Delta \\ &= \frac{5EI}{6} \theta_B^2 + \frac{EI}{3} \theta_B \theta_C + \frac{17EI}{24} \theta_C^2 - 40\theta_B + 24\theta_C \end{aligned}$$

In terms of the principle of potential energy, the equilibrium equations of the displacement method can be obtained:

$$\frac{\partial \Pi}{\partial \theta_B} = \frac{5EI}{3} \theta_B + \frac{EI}{3} \theta_C - 40 = 0$$

$$\frac{\partial \Pi}{\partial \theta_C} = \frac{EI}{3} \theta_B + \frac{17EI}{12} \theta_C + 24 = 0$$

Solving the above equations simultaneously yields

$$\theta_B = \frac{776}{27EI}, \quad \theta_C = -\frac{640}{27EI}$$

6.8 Rayleigh–Ritz Method

One of the important applications of the principle of potential energy is the approximate analysis of structures that cannot be solved accurately or are very difficult to solve. This approximate analysis method is the **Rayleigh–Ritz method**. Applying the principle of potential energy, it is necessary to express the strain energy U as a function of the joint displacements, which is often quite difficult. In this case, the true deformation curve can be approximated by a hypothetical deformation curve. This kind of deformation curve can be represented by a displacement function, which contains one or more indefinite displacement parameters, and then the potential energy can be expressed as a function of the above-mentioned displacement parameters. According to the principle of potential energy, the partial derivative of potential energy with respect to each displacement parameter should be zero, from which a set of equations expressed by the unknown displacement parameters can be obtained. Solving the equations simultaneously yields the

displacement parameters, and then the hypothetical deformation curve can be obtained. Based on this, the internal forces can be further determined. This is the main idea of the Rayleigh–Ritz method for structural analysis.

Example 6.9. Determine the approximate value of the deflection at the center of the stepped beam shown in figure 6.31.

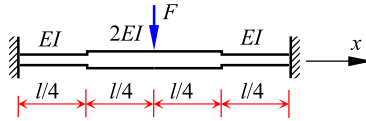


FIG. 6.31 – A stepped beam.

Solution:

The precise calculation of the beam displacement is complicated, but the Rayleigh–Ritz method can be used to find its approximate value more conveniently. Assuming the displacement function of the deflection curve is

$$y = A_1 x^3 + A_2 x^2 + A_3 x + A_4 \quad \left(0 \leq x \leq \frac{l}{2}\right) \quad (6.27)$$

Regarding the undetermined constants (*i.e.*, the displacement parameters) in equation (6.27), the following conditions should be satisfied: (1) when $x = 0$, $y = 0$, then $A_4 = 0$; (2) when $x = 0$, $y' = 0$, then $A_3 = 0$; (3) when $x = \frac{l}{2}$, from the symmetry, $y' = 0$, then $A_2 = -\frac{3A_1 l}{4}$; (4) when $x = \frac{l}{2}$, assuming $y = \Delta$, then $A_1 = -\frac{16\Delta}{l^3}$. Substituting the above results into equation (6.27), the displacement function expression can be obtained,

$$y = \frac{4\Delta x^2}{l^3}(3l - 4x) \quad \left(0 \leq x \leq \frac{l}{2}\right) \quad (6.28)$$

For the segmented straight beams with constant section, the bending strain energy of the full beam is

$$U = \sum \int \frac{M^2(x)}{2EI} dx$$

According to the approximate differential equation of the deflection curve $EIy'' = M(x)$, we have

$$U = \sum \frac{EI}{2} \int (y'')^2 dx$$

For the stepped beam in this example, the bending strain energy is

$$U = 2 \left[\frac{EI}{2} \int_0^{l/4} (y'')^2 dx + EI \int_{l/4}^{l/2} (y'')^2 dx \right]$$

Substituting equation (6.28) into it and integrating, we get

$$U = \frac{144EI\Delta^2}{l^3}$$

Then, the total potential energy of the beam is

$$\Pi = U - F\Delta = \frac{144EI\Delta^2}{l^3} - F\Delta$$

Applying the principle of minimum potential energy, we obtain

$$\frac{d\Pi}{d\Delta} = \frac{288EI\Delta}{l^3} - F = 0$$

Thus, we get

$$\Delta = \frac{Fl^3}{288EI} = 0.00347 \times \frac{Fl^3}{EI}$$

The precise result of central deflection of this beam is

$$\Delta = \frac{11Fl^3}{3072EI} = 0.00358 \times \frac{Fl^3}{EI}$$

Therefore, the approximate result is only 3% smaller than the precise result. It should be pointed out:

- (1) The appropriate selection of displacement function is very important to ensure the accuracy of the final result. Therefore, the selected displacement function should be close to the actual deformation curve. The more precise the selected displacement function is, the higher the accuracy of the calculation result. As a minimum requirement, the choice of displacement function should satisfy the displacement boundary conditions of the structure. For a beam, that is to meet the conditions of deflection and rotation.
- (2) Since the calculated displacement is approximate, the reaction forces and internal forces represented by the derivative of the displacement function are also approximate, so the equilibrium condition of the structure can only be approximately satisfied. The Rayleigh–Ritz method can not only approximately solve the displacement and internal force of structures, but also approximately calculate the instability critical pressure and natural frequency of structures, etc.

6.9 Moment-Distribution Method and No-Shear Distribution Method

To calculate a statically indeterminate frame, regardless of either the force method or the displacement method used, the canonical equations must be established and solved. When there are many unknowns, the workload of solving simultaneous

equations is very heavy. In order to find a simpler way to calculate statically indeterminate frames, various asymptotic displacement methods have appeared since the 1930s, such as the **moment-distribution method**, the **no-shear distribution method**, and the moment iteration method. These methods are variants of the displacement method. Their common feature is to avoid the establishment and solution of canonical equations but to use the asymptotic method to calculate the member end moments. The accuracy of the results increases as the number of calculation rounds increases, and finally converges to an accurate solution. The physical concepts of these methods are vivid, and each round of calculation is repeated in the same step. Thus, it is easy to grasp, suitable for hand calculation, and can directly obtain the member end moments without calculating the joint displacements. Therefore, in the past, they have been widely applied in the structural analysis and design of frames, multi-span bridges, roofs and floors and continuous beams, etc. With the popularization of computers and the matrix displacement method programs, the application of this kind of hand calculation method has decreased, but it is still a simple and easy method for preliminary analysis and design when the structures are a few unknowns.

6.9.1 Basic Principle of the Moment-Distribution Method

The moment-distribution method is particularly convenient for the calculation of continuous beams and frames without sidesway.

The concepts of rotational stiffness and carryover factor are used in the moment-distribution method, which are defined as follows: when end A of member AB (figure 6.32) rotates by a unit angle, the moment M_{AB} of end A (also called the near end) is called the rotational stiffness of the member end and is denoted as S_{AB} . It represents the resistance of the member end to rotation. Its value is not only related to the linear stiffness of the member, but also related to the support of another end (also called the far end) of the member. When end A rotates, end B also generates a certain moment, which is like the moment of the near end carried over to the far end in some proportion. Thus, the ratio of the moment of end B to that of end A is called the carryover factor from end A to end B, which is expressed by C_{AB} , that is $C_{AB} = \frac{M_{BA}}{M_{AB}}$, or $M_{BA} = C_{AB}M_{AB}$. Table 6.3 lists the rotational stiffness and carryover factor of prismatic members. When end B is free or has an axial support link, obviously the member will have no resistance when end A rotates, so its rotational stiffness is zero.

The moment-distribution method is a structural analysis method evolved from the displacement method, so the sign convention of angular displacements of the joints and the member end forces are the same as that of the displacement method.

Now the frame shown in figure 6.33a is taken as an example to illustrate the basic principle of the moment-distribution method. When the frame is calculated by the displacement method, there is only one primary unknown, namely the joint rotational angle X_1 , and its canonical equation is that of the displacement method, and the symbol should be consistent with section 6.6.

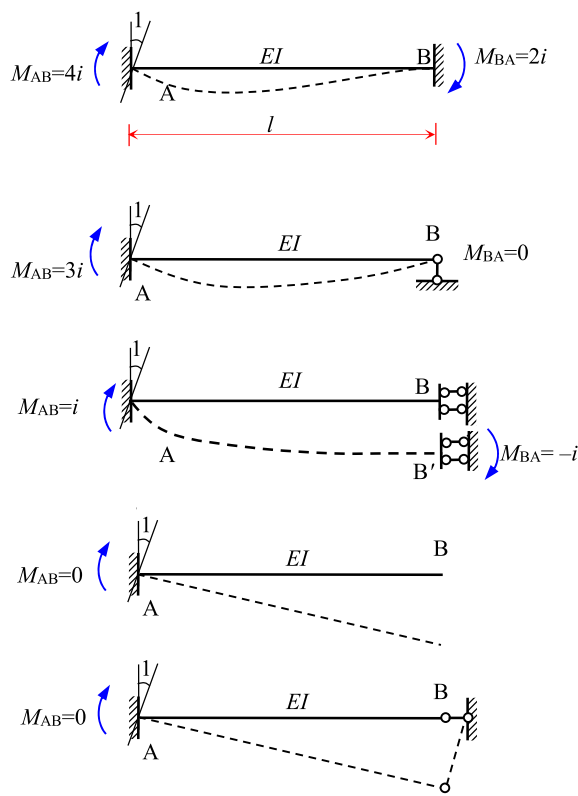


FIG. 6.32 – Rotational stiffnesses of prismatic members.

TAB. 6.3 – Rotational stiffnesses and carryover factors of prismatic members

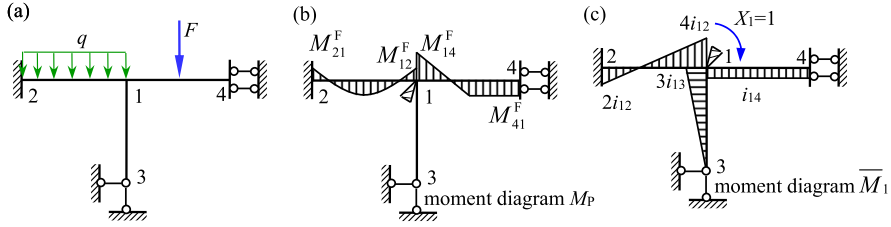
Support condition at far end	Rotational stiffness S	Carryover factor C
Fixed support	$4 i$	0.5
Pin support	$3 i$	0
Slider support	i	-1
Free or axial strut	0	

$$k_{11}X_1 + F_{1P} = 0$$

Drawing the diagrams M_P and $\overline{M}_1$ (figure 6.33b, c), the free term can be obtained as

$$F_{1P} = M_{12}^F + M_{13}^F + M_{14}^F = \sum M_{1j}^F \tag{6.29}$$

F_{1P} is the reaction moment on the additional rigid arm when the joint is fixed, which is called the reaction moment of rigid arm. It is equal to the algebra sum

FIG. 6.33 – Structural analysis of frame *via* displacement method.

$\sum M_{1j}^F$ of the fixed-end moment of each member that meets at joint 1, that is, the difference that cannot be balanced by the fixed-end moment of each member, so it is also called the unbalanced moment of joint.

$$k_{11} = 4i_{12} + 3i_{13} + i_{14} = S_{12} + S_{13} + S_{14} = \sum S_{1j} \quad (6.30)$$

where, $\sum S_{1j}$ represents the sum of the rotational stiffness of each member end meeting at joint 1.

Solving the canonical equation yields

$$X_1 = -\frac{F_{1P}}{k_{11}} = \frac{-\sum M_{1j}^F}{\sum S_{1j}}$$

The final end moment of each member can be calculated according to the superposition principle $M = M_P + \bar{M}_1 X_1$. The end where each member meets at joint 1 is the near-end, and the other end is the far end. The near end moments are

$$\left. \begin{aligned} M_{12} &= M_{12}^F + \frac{S_{12}}{\sum S_{1j}} \left(-\sum M_{1j}^F \right) = M_{12}^F + \mu_{12} \left(-\sum M_{1j}^F \right) \\ M_{13} &= M_{13}^F + \frac{S_{13}}{\sum S_{1j}} \left(-\sum M_{1j}^F \right) = M_{13}^F + \mu_{13} \left(-\sum M_{1j}^F \right) \\ M_{14} &= M_{14}^F + \frac{S_{14}}{\sum S_{1j}} \left(-\sum M_{1j}^F \right) = M_{14}^F + \mu_{14} \left(-\sum M_{1j}^F \right) \end{aligned} \right\} \quad (6.31)$$

The first term on the right side of the above equations is the moment generated by the load, that is, the fixed-end moment. The second term is the moment generated by the joint rotational angle X_1 , which is equivalent to multiplying the unbalanced moment by the minus sign, and then distributing it to the near ends according to the ratio of the stiffness coefficient, so it is called the distributed moment. μ_{12} , μ_{13} , μ_{14} are called the distribution factors. Their calculation formula is

$$\mu_{1j} = \frac{S_{1j}}{\sum S_{1j}} \quad (6.32)$$

Obviously, the sum of the distribution factors of the member ends of the same joint should be equal to 1, that is, $\sum \mu_{1j} = 1$.

The far end moments are written as

$$\left. \begin{aligned} M_{21} &= M_{21}^F + \frac{C_{12}S_{12}}{\sum S_{1j}} \left(-\sum M_{1j}^F \right) = M_{21}^F + C_{12} \left[\mu_{12} \left(-\sum M_{1j}^F \right) \right] \\ M_{31} &= M_{31}^F + C_{13} \left[\mu_{13} \left(-\sum M_{1j}^F \right) \right] \\ M_{41} &= M_{41}^F + C_{14} \left[\mu_{14} \left(-\sum M_{1j}^F \right) \right] \end{aligned} \right\} \quad (6.33)$$

The first term on the right side of each formula is still the fixed-end moment. The second term is the moment generated by the joint rotational angle X_1 , which is like carrying over the distributed moment of near end to far end in the proportion of the carryover factor, so it is called the carryover moment.

According to the above-mentioned rules, the moment diagrams of M_P and $\overline{M}_1$ do not need to be drawn. We do not need to write and solve the canonical equations, while directly calculate each member end moment according to the above conclusions. The process can be visually summarized as two steps:

- (1) Fix the joints. Add the rigid arm, then each member end has a fixed-end moment, and there is an unbalanced moment on the joint, which is temporarily borne by the rigid arm.
- (2) Relax the joints. Remove the rigid arm and let the joint rotate. This is equivalent to adding an opposite unbalanced moment to the joint, so the unbalanced moment is eliminated and the joint is balanced. This opposite unbalanced moment will be distributed to the near ends in proportion to the rotational stiffness so that each near end gets the assigned moment, which carries over to the far end.

Finally, each near-end moment is equal to the fixed-end moment plus the distributed moment, and each far-end moment is equal to the fixed-end moment plus the carryover moment.

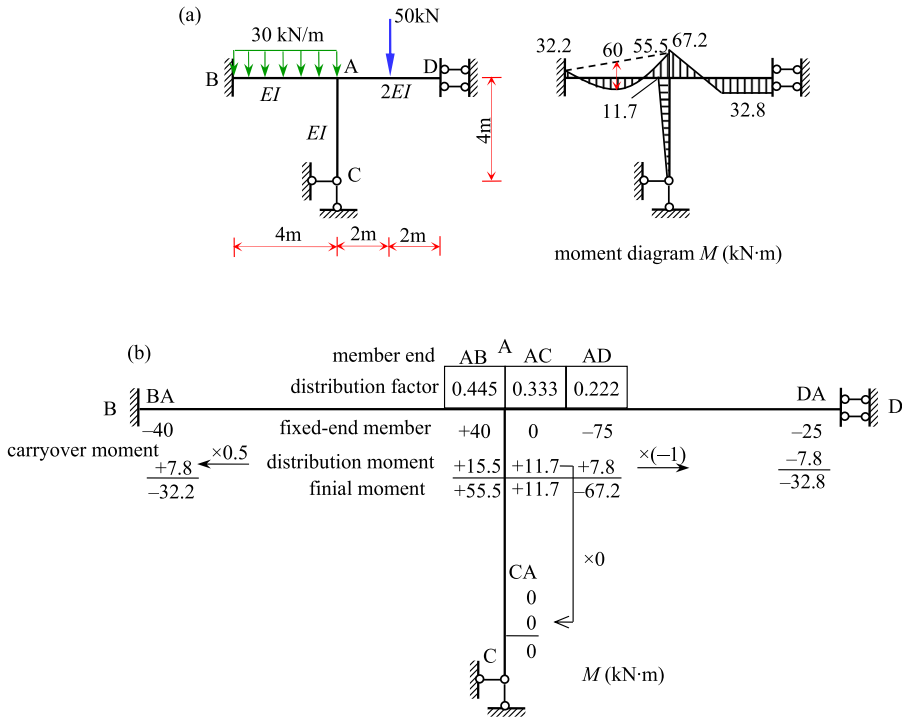
Example 6.10. Draw the moment diagram of the frame shown in figure 6.34a.

Solution: (1) Calculate the distribution factors of member ends. For the convenience of calculation, let $i_{AB} = i_{AC} = \frac{EI}{4} = 1$, then $i_{AD} = 2$. From equation (6.32), the distribution factors are calculated

$$\mu_{AB} = \frac{4 \times 1}{4 \times 1 + 3 \times 1 + 2} = \frac{4}{4 + 3 + 2} = \frac{4}{9} = 0.445$$

$$\mu_{AC} = \frac{3}{9} = 0.333$$

$$\mu_{AD} = \frac{2}{9} = 0.222$$

FIG. 6.34 – Structural analysis of frame *via* moment-distribution method.

- (2) Calculate the moments of fixed ends. According to table 6.2, we have

$$M_{BA}^F = -\frac{30 \text{ kN/m} \times (4 \text{ m})^2}{12} = -40 \text{ kN} \cdot \text{m}$$

$$M_{AB}^F = +\frac{30 \text{ kN/m} \times (4 \text{ m})^2}{12} = +40 \text{ kN} \cdot \text{m}$$

$$M_{AD}^F = -\frac{3 \times 50 \text{ kN/m} \times 4 \text{ m}}{8} = -75 \text{ kN} \cdot \text{m}$$

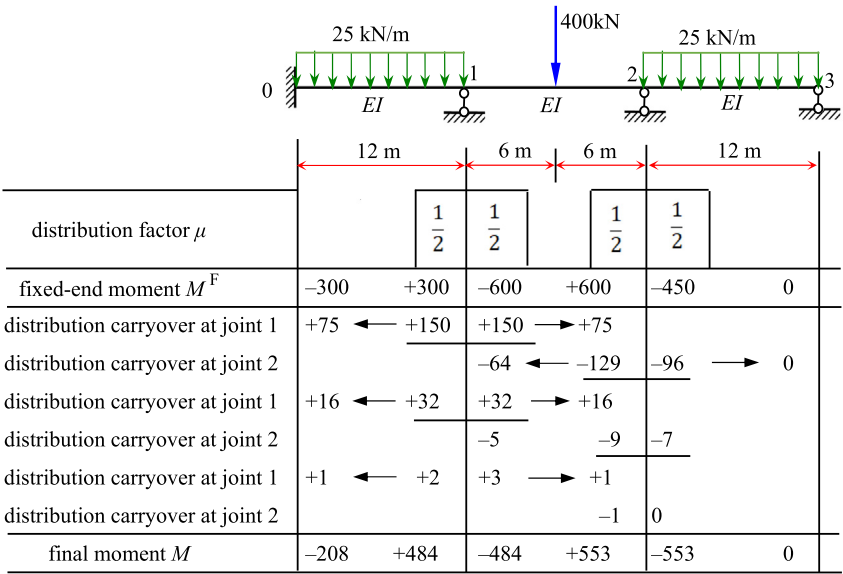
$$M_{DA}^F = -\frac{50 \text{ kN/m} \times 4 \text{ m}}{8} = -25 \text{ kN} \cdot \text{m}$$

- (3) Calculate the distributed and carryover moments. The unbalanced moment of joint A is $\sum M_{Aj}^F = 40 - 75 = -35 \text{ kN} \cdot \text{m}$, and the opposite unbalanced moment is multiplied by the distribution factor to obtain the distributed moment of each near end, and then the carryover factor is multiplied to obtain the carryover moment of each far end. In the moment-distribution method, in order to make the expression of the calculation process more compact and intuitive, and avoid writing a large number of calculation formulas, the entire calculation can be directly written on the graph (or tabular calculation), as shown in figure 6.34b.

- (4) Calculate the final member end moments. The final end moment of each member is obtained by superimposing the fixed-end moment, the distributed moment, and the carryover moment. Based on this, the moment diagram of the frame can be drawn, as shown in figure 6.34c.

6.9.2 Computation of Continuous Beams and Frames Without Sidesway Using the Moment-Distribution Method

The basic principle of the moment-distribution method has been elaborated through a structure with only one joint rotation in example 6.10. For a structure with multiple joint angular displacements but no joint linear displacements (abbreviated as no sidesway), it can be solved by using the method described in section 6.9.2 for each joint in turn. The procedure is presented as follows: firstly, fix all the joints and calculate the fixed-end moment of each member; then relax each joint in turn, that is, only relax one joint at a time, and the other joints are still temporarily fixed. The unbalanced moment is distributed and carried over in turn until the carryover moment is so small that can be omitted. The member end moment is calculated by such a successive asymptotic method. We illustrate this with specific examples.



Note: The unit of moment is kN·m

FIG. 6.35 – Analysis of continuous beam *via* moment-distribution method.

The continuous beam shown in figure 6.35 has two joint angular displacements and no joint linear displacement. Now that the two rigid joints 1 and 2 are fixed, the moment of each member can be calculated as

$$\begin{aligned}
M_{01}^F &= -\frac{25\text{kN/m} \times (12\text{ m})^2}{12} = -300\text{ kN}\cdot\text{m} \\
M_{10}^F &= +\frac{25\text{ kN/m} \times (12\text{ m})^2}{12} = +300\text{ kN}\cdot\text{m} \\
M_{12}^F &= -\frac{400\text{kN/m} \times 12\text{ m}}{8} = -600\text{ kN}\cdot\text{m} \\
M_{21}^F &= +\frac{400\text{kN/m} \times 12\text{ m}}{8} = +600\text{ kN}\cdot\text{m} \\
M_{23}^F &= -\frac{25\text{kN/m} \times (12\text{ m})^2}{8} = -450\text{ kN}\cdot\text{m} \\
M_{32}^F &= 0
\end{aligned}$$

The above values are filled into the column of the fixed-end moment M^F in figure 6.35. There are unbalanced moments on joints 1 and 2

$$\begin{aligned}
\sum M_{1j}^F &= +300\text{ kN}\cdot\text{m} - 600\text{ kN}\cdot\text{m} = -300\text{ kN}\cdot\text{m} \\
\sum M_{2j}^F &= +600\text{ kN}\cdot\text{m} - 450\text{ kN}\cdot\text{m} = +150\text{ kN}\cdot\text{m}
\end{aligned}$$

To eliminate these two unbalanced moments, in the displacement method, joints 1 and 2 are made to produce the same rotational angle as the original structure, that is, the two joints are relaxed at the same time, and they are rotated around the actual balance position meanwhile. As mentioned earlier, this requires establishing simultaneous equations and solving them. In the moment-distribution method, the situation is different: the joints are gradually relaxed in turn to achieve the same goal.

At first, relax joint 1, and joint 2 is still fixed, so it is exactly the same as the situation of relaxing a single joint in the previous section. Therefore, the unbalanced moment of joint 1 can be eliminated according to the aforementioned method of moment distributed and carried over. For this reason, the distribution factors of member end at joint 1 need to be obtained first. Since each span EI and l are the same, the linear stiffness is i , and according to equation (6.32), we get

$$\mu_{10} = \frac{4i}{4i + 4i} = \frac{1}{2}, \quad \mu_{12} = \frac{4i}{4i + 4i} = \frac{1}{2}$$

Fill it into the column of the distribution factor μ in figure 6.35. The unbalanced moment $-300\text{ kN}\cdot\text{m}$ of joint 1 is first reversed and then distributed, and the distributed moment can be obtained as

$$\begin{aligned}
M_{10} &= \frac{1}{2} \times [-(-300\text{ kN}\cdot\text{m})] = +150\text{ kN}\cdot\text{m} \\
M_{12} &= \frac{1}{2} \times [-(-300\text{ kN}\cdot\text{m})] = +150\text{ kN}\cdot\text{m}
\end{aligned}$$

Fill them into the figure. In this way, joint 1 is temporarily balanced, and a horizontal line is drawn under the distributed moment to indicate the balance. At this time, joint 1 has also rotated an angle (but has not yet moved to the final position). Meanwhile, the distributed moment should be carried over to the respective far ends, and the carryover moments are

$$M_{01} = \frac{1}{2} \times (+150 \text{ kN}\cdot\text{m}) = +75 \text{ kN}\cdot\text{m}$$

$$M_{21} = \frac{1}{2} \times (+150 \text{ kN}\cdot\text{m}) = +75 \text{ kN}\cdot\text{m}$$

In the figure, the arrows represent the moment carried over to the far end.

Secondly, for joint 2, its original unbalanced moment is $+150 \text{ kN}\cdot\text{m}$, plus the carryover moment $+75 \text{ kN}\cdot\text{m}$ from joint 1, so the total unbalanced moment is $+150 \text{ kN}\cdot\text{m} + 75 \text{ kN}\cdot\text{m} = +225 \text{ kN}\cdot\text{m}$. Reset the rigid arm to fix joint 1 at the position after the rotation just now, and then relax joint 2, which is the same as the situation of relaxing a single joint in the previous section. The member end distribution factors of joint 2 are

$$\mu_{21} = \frac{4i}{4i+3i} = \frac{4}{7}, \quad \mu_{23} = \frac{3i}{4i+3i} = \frac{3}{7}$$

Reverse the unbalanced moment $+225 \text{ kN}\cdot\text{m}$ and distribute it:

$$M_{21} = \frac{4}{7} \times (-225 \text{ kN}\cdot\text{m}) = -129 \text{ kN}\cdot\text{m}$$

$$M_{23} = \frac{3}{7} \times (-225 \text{ kN}\cdot\text{m}) = -96 \text{ kN}\cdot\text{m}$$

Simultaneously carry over them to the corresponding far ends:

$$M_{12} = \frac{1}{2} \times (-129 \text{ kN}\cdot\text{m}) = -64 \text{ kN}\cdot\text{m}$$

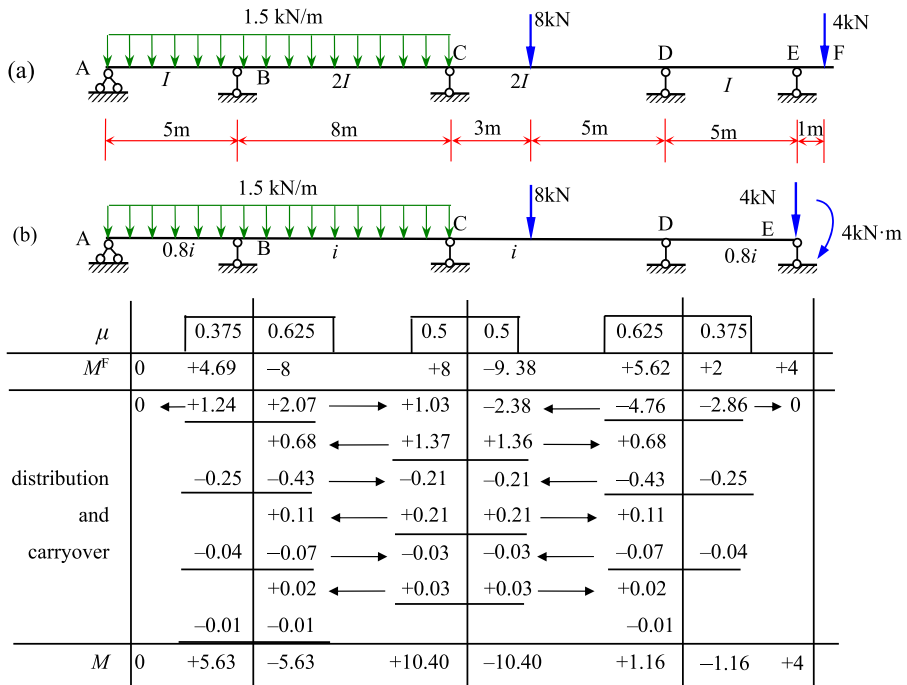
$$M_{32} = 0 \times (-96 \text{ kN}\cdot\text{m}) = 0 \text{ kN}\cdot\text{m}$$

Therefore, joint 2 is temporarily balanced, and at the same time it is rotated by an angle (not to the final position), and then it is fixed again at the rotated position.

Look at joint 1 again, it has a new unbalanced moment $-64 \text{ kN}\cdot\text{m}$, so joint 1 is relaxed again, and the distribution and carryover are implemented out in the same way, and so on. In this way, each joint is fixed and relaxed repeatedly, and the moment is continuously distributed and carried over. The value of the unbalanced moment will become smaller and smaller (because the distribution factor and carryover factor are both less than 1), until the carryover moment value is small enough to be omitted according to the calculation accuracy requirements, and the calculation can be stopped. At this time, after each joint rotates successively, it gradually approaches its final equilibrium position.

The final end moment of each member is obtained by adding up the fixed-end moment of each member and the distributed and carryover moments obtained repeatedly.

Example 6.11. Apply the moment-distribution method to analyze the continuous beam exhibited in figure 6.36a and draw the moment diagram.



Note: The unit of moment is kN·m

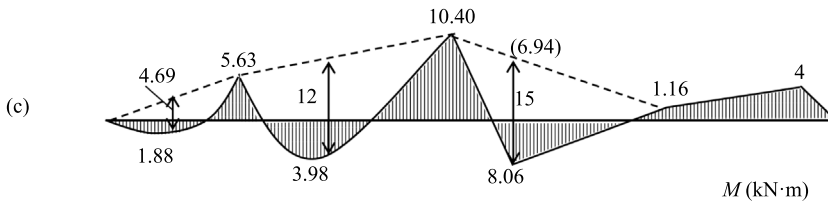


FIG. 6.36 – Analysis of continuous beam by using moment-distribution method.

Solution: (1) The cantilever part EF on the right is statically determined. If it is cut off, and the corresponding moments and shears are applied to joint E as the external force, then joint E will be turned into a pinned end, as shown in figure 6.36b.

(2) Calculate the distribution factors. If the linear stiffness of the members BC and CD is $\frac{2EI}{8} = i$, then the linear stiffness of the members AB and DE is converted to $\frac{EI}{5} = 0.8i$, as shown in figure 6.36b. For joint D, the distribution factors are

$$\mu_{DC} = \frac{4i}{4i + 3 \times 0.8i} = \frac{4}{4 + 2.4} = 0.625$$

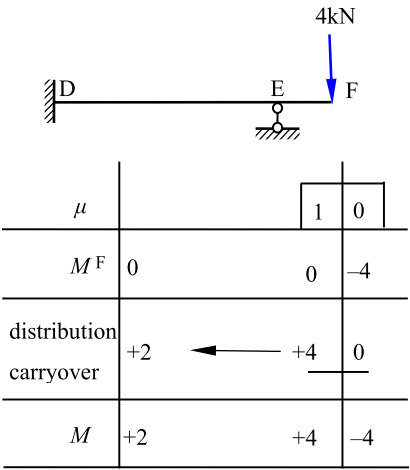
$$\mu_{DE} = \frac{2.4}{4 + 2.4} = 0.375$$

The distribution factors of the remaining joints can be calculated in the same way, as shown in the figure.

(3) Calculate the fixed-end moments. The member DE is equivalent to a beam with one end fixed and the other end pinned, bearing a concentrated force and a couple at the pinned end. Among them, the concentrated force of 4 kN will be directly borne by the support E without causing the beam to generate moment, so it is not considered; while the fixed-end moment generated by the couple of 4 kN·m can be calculated from table 6.2

$$M_{DE}^F = \frac{1}{2} \times 4 \text{ kN} \cdot \text{m} = +2 \text{ kN} \cdot \text{m}$$
$$M_{ED}^F = +4 \text{ kN} \cdot \text{m}$$

In addition, the fixed-end moment of the member DE can also be obtained using the moment-distribution method. As shown in figure 6.37, there is no need to remove the cantilever, but the joint E is also temporarily fixed, so each fixed-end moment can be written in the figure. Then, relax the joint E. Since member EF is a cantilever beam, the rotational stiffness of E end is zero, so it is known that the distribution factors are $\mu_{EF} = 0$, and $\mu_{ED} = 1$. Therefore, the unbalanced moment of joint E will be all distributed to end E of the member DE after the reverse sign, and half of it will be carried over to end D. The calculation is shown in figure 6.37, and the result is the same as above. After joint E is relaxed this time, it will not be fixed again, and will be treated as a pinned end in subsequent calculations.



Note: The unit of moment is kN·m

FIG. 6.37 – Analysis of cantilever beam *via* moment-distribution method.

The remaining moments of each fixed end can be obtained according to table 6.2, and there is no need to describe the details.

(4) Take turns to relax each joint for moment distribution and carryover. In order to make the calculation faster, the distribution should start from the joint with the larger value of the unbalanced moment. In this example, first relax joint D. In addition, since joint C is fixed when joint D is relaxed, joint B can be relaxed at the same time. It can be

seen that all non-adjacent joints can relax at the same time so that the speed of convergence can be accelerated. The whole calculation process is shown in figure 6.36b.

(5) Calculate the final member end moments and draw the M diagram (figure 6.36c).

Example 6.12. Utilize the moment-distribution method to draw the moment diagram of the frame displayed in figure 6.38a.

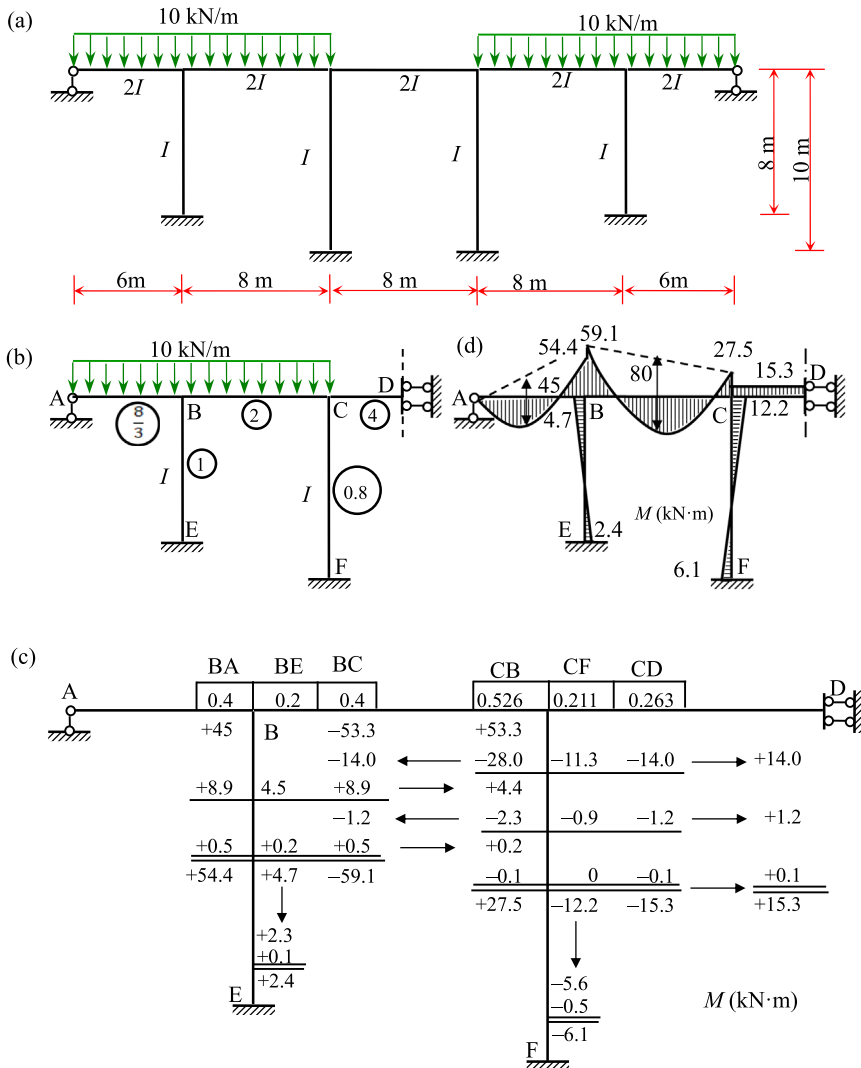


FIG. 6.38 – Analysis of symmetric frame subjected to symmetric loads *via* moment-distribution method.

Solution: This is a symmetric frame subjected to symmetric loads. The half structure is shown in figure 6.38b. There are two joint angular displacements and no joint linear displacement (without sidesway). For convenience, assuming $\frac{EI}{8} = 1$, and the linear stiffness of each member can be calculated as denoted by the small circle in the figure. All other calculations are shown in figure 6.38c, without detailed description. After the calculation is completed, it can be checked whether the member end moments at each joint meet the equilibrium condition. For joint B, it has

$$\sum M_{Bj} = +54.4 \text{ kN}\cdot\text{m} + 4.7 \text{ kN}\cdot\text{m} - 59.1 \text{ kN}\cdot\text{m} = 0$$

Joint C has

$$\sum M_{Cj} = +27.5 \text{ kN}\cdot\text{m} - 12.2 \text{ kN}\cdot\text{m} - 15.3 \text{ kN}\cdot\text{m} = 0$$

Therefore, the calculation is correct. The final diagram M is shown in figure 6.38d.

6.9.3 No-Shear Distribution Method

The no-shear distribution method is a calculation method for frames with sidesway that meet certain conditions. This section takes the half-structure of a single-span symmetric frame under antisymmetric load as an example to illustrate this method.

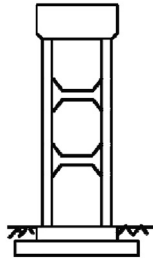


FIG. 6.39 – Frame-type pier.

Single-span symmetric frames are common in engineering, such as frame-type pier (figure 6.39), brace of aqueduct or pipeline, and single-span plant. For the single-span symmetric frame shown in figure 6.40a, the load can be divided into two groups: symmetric and antisymmetric load. In the case of a symmetric load (figure 6.40b), the joints have only angular displacement. Therefore, it can be calculated by the aforementioned general moment-distribution method, and it is not discussed here. When it is an antisymmetric load (figure 6.40c), in addition to the rotation, the joint also has a linear displacement. In this case, the following no-shear distribution method can be used to calculate.

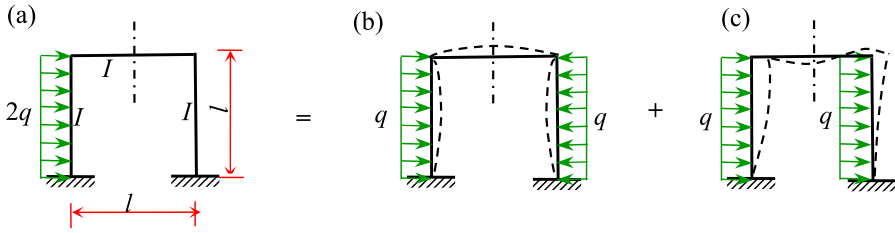


FIG. 6.40 – (a) Single-span symmetric frame, (b) symmetric load, and (c) antisymmetric load.

The halfframe under antisymmetric load is shown in figure 6.41a, and C is a roller support. The deformation and force of the half frame have the following characteristics: member BC has a horizontal displacement but there is no relative linear displacement between its two ends, so it is called a non-sway movement member; Column AB has a linear displacement between its two ends because there is no horizontal reaction force at the support C, the shear of column AB is statically determined, which is called a shear statically determinate member. When calculating this half-frame, it can still be considered in two steps like the moment distribution method:

- (1) Fix the joint. A rigid arm is only added to restrain the rotation of joint B, without adding a link to prevent its linear displacement, as shown in figure 6.41b. In this way, although the upper end of column AB cannot rotate, it can still slide freely horizontally, and it is equivalent to the beam with the lower end fixed and the upper end slider-supported (figure 6.41c). For the crossbeam BC, as its horizontal movement does not affect its internal force, it is equivalent to a beam with one end fixed and the other end pin supported. From table 6.2, it can be found that the moments at the fixed end of the columns are

$$M_{AB}^F = -\frac{ql^2}{3}, \quad M_{BA}^F = -\frac{ql^2}{6} \quad (6.34)$$

The unbalanced moment at joint B is temporarily borne by the rigid arm. Note that the shear of column AB is still statically determinate at this time, and the shears at both ends are

$$F_{SBA} = 0, \quad F_{SAB} = ql \quad (6.35)$$

That is, all horizontal loads are balanced by the shears at the lower end of the columns.

- (2) Relax the joint. To eliminate the unbalanced moment on the rigid arm, the joint is now relaxed and the moment is distributed and carried over. At this time, joint B not only rotates an angle X_1 but also has horizontal displacement, as shown in figure 6.41d. Since the lower end of column AB is fixed and the upper end is slider-supported, the shear of the column is zero when the upper end rotates, so it is in a pure bending state (figure 6.41e). The forces and

deformation state (figure 6.41f) are exactly the same, so it can be inferred that the rotational stiffness should be i , and the carryover factor should be -1 .

Therefore, the distribution factors of joint B are

$$\mu_{BA} = \frac{i}{i + 3 \times 2i} = \frac{1}{7}, \quad \mu_{BC} = \frac{3 \times 2i}{i + 3 \times 2i} = \frac{6}{7} \quad (6.36)$$

The rest of the calculation is shown in figure 6.42, and the details are omitted. The moment diagram M is exhibited in figure 6.41g.

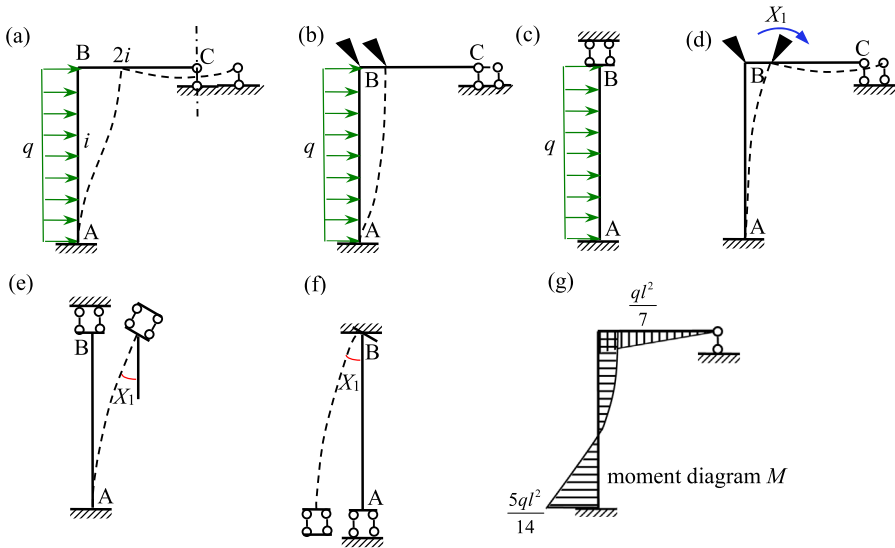


FIG. 6.41 – Analysis of half frame under antisymmetric load by no-shear distribution method.

It is known that the shear of column AB is statically determinate when the joint is fixed; when the joint is relaxed, the distributed moment obtained at end B of column AB should be multiplied by the carryover factor of -1 and carried over to end A, so the moment remains unchanged along the full length of member AB and the shears is zero. In this way, in the process of moment distribution and carryover, the original shears in the column will remain unchanged, and no new shears will be added. Therefore, this method is called the no-shear moment distribution method, or simply the no-shear distribution method.

The above method can be extended to multi-story situations. As shown in figure 6.43a, each beam is a no-sidesway member, and each column is a shear determinate member. When the rigid arms are added to fix the joints, they only restrain the rotation of each joint, but do not restrain their linear displacements, as shown in figure 6.43b. At this time, there are no rotations at both ends of each story of columns, but there are linear displacements. When investigating the relative lateral movement of the two ends of any column, such as column BC, the lower end is considered fixed and the upper end is sliding. From the equilibrium conditions, the

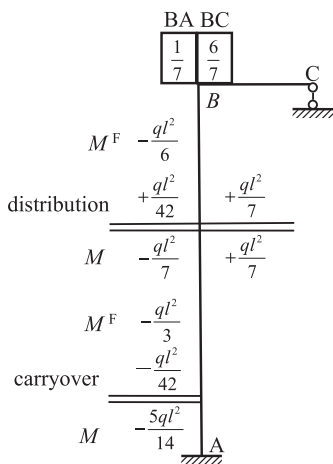


FIG. 6.42 – Distribution and carryover factors of moments of half frame.

shear at the upper end can be determined (figure 6.43c). It can be inferred that no matter how many stories the frame has, the column of each story can be regarded as a beam with the upper end slider-supported and the lower end fixed. In addition to bearing the load of this story, the upper ends of the columns also bear the shears, which are equal to the algebraic sum of all horizontal loads on each story above it. In this way, the fixed-end moments of the columns of each story can be calculated according to table 6.2. Then, relax each joint, in turn, to distribute and carry over the moment. Figure 6.43d shows the situation when joint C is relaxed, which is equivalent to applying the opposite of the unbalanced moment on the joint as a couple of loads on the joint. In this case, not only joint C rotates at a certain angle θ_C , but the two columns BC and CD will also have relative side shifts. However, it is

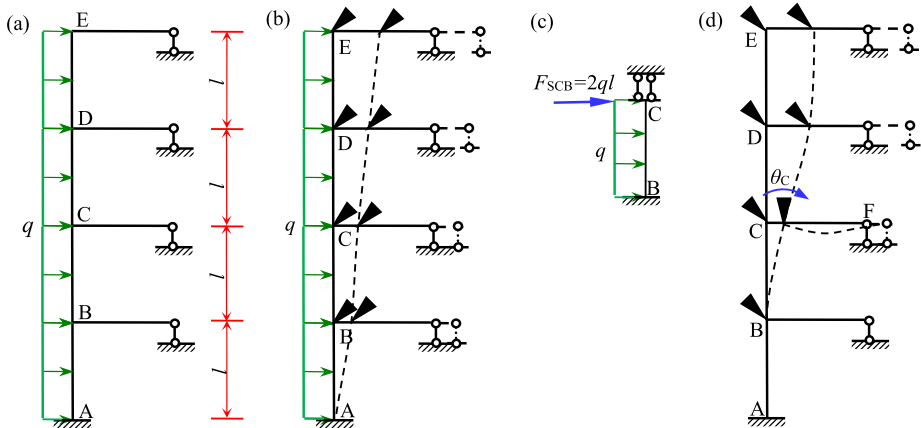


FIG. 6.43 – No-shear distribution method for multi-story frame analysis.

known from the equilibrium condition that the shears of the two columns are zero, and they are in a state of pure bending (as shown in figure 6.41f), so the rotational stiffness of each column should be calculated as its respective linear stiffness i , and the carryover factor is -1 (referring to the constant cross-section member). It is worth pointing out that, only the members meeting at joint C will deform and receive force; the stories below joint B have no displacement and therefore are not stressed; the stories above joint D will move horizontally along with joint D, but there is no sidesway between the ends of the members, so they are still not stressed. Therefore, when joint C is relaxed, the distribution and carryover of the moment only need to be carried out within the range of members CB, CF, and CD. The situation is similar when relaxing other joints. The specific calculation steps for distributed and carryover moments are the same as the general moment-distribution method, so we omit the details.

The no-shear distribution method is used to calculate the frame with sidesway. Because of the special measures that only control the rotation of the joint and allow it to have linear displacements, the calculation process is as simple as the moment distribution method. However, it should be noted that the no-shear distribution method is only applicable to some special frames with sidesway, namely, a part of the members in the frame has no lateral movement, and the rest are statically determinate shear members. For example, there is only one column and the support bars at the outer end of each member are parallel to the column (figure 6.44).

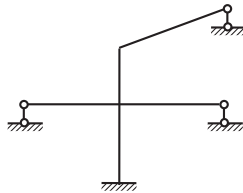


FIG. 6.44 – Special frame with sidesway.

Example 6.13. Apply the no-shear distribution method to construct the moment diagram of the frame shown in figure 6.45a.

Solution: When calculating the distribution factors, the end rotational stiffness of each column should be equal to the linear stiffness of the column. According to table 6.2, the fixed-end moments can be calculated, for the AC column there are

$$M_{AC}^F = -\frac{10 \text{ kN} \times 4 \text{ m}}{8} = -5 \text{ kN} \cdot \text{m}$$

$$M_{CA}^F = -\frac{3 \times 10 \text{ kN} \times 4 \text{ m}}{8} = -15 \text{ kN} \cdot \text{m}$$

For column CE, in addition to a load of this story, it is also subjected to the shear 10 kN at the column top, so the fixed-end moments can be obtained

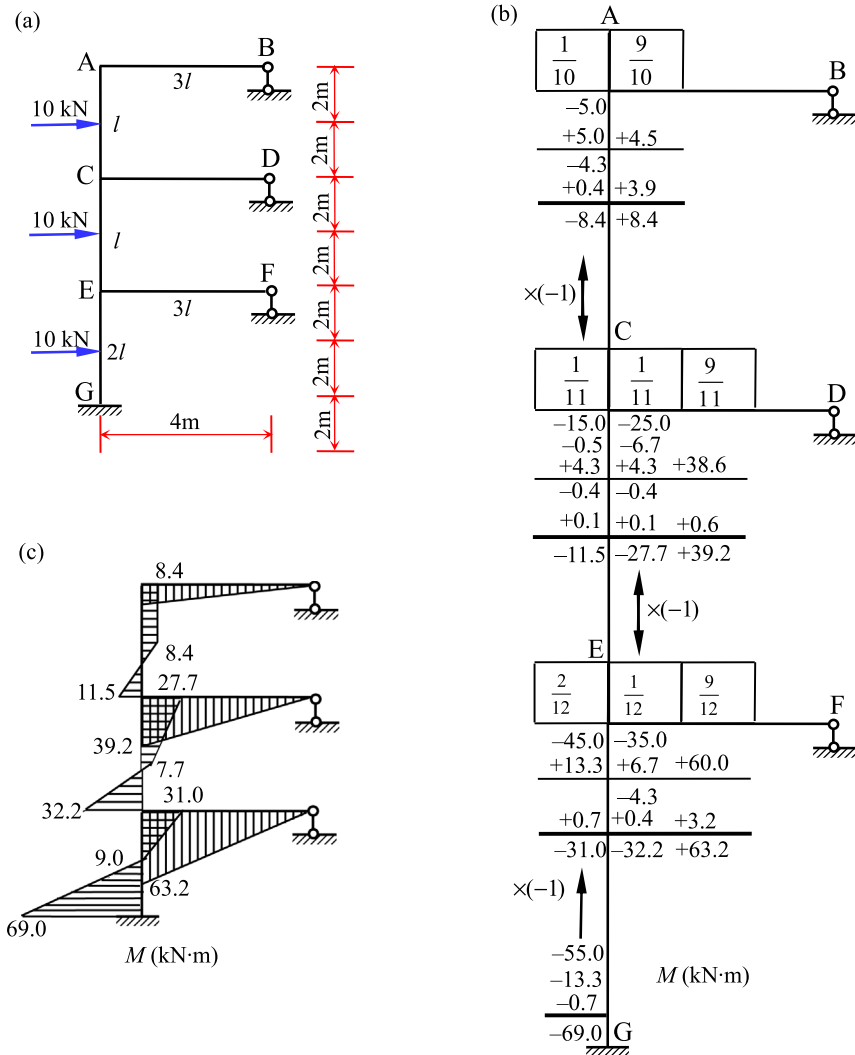


FIG. 6.45 – Analysis of multi-story frame with sidesway by no-shear distribution method.

$$\begin{aligned} M_{CE}^F &= -\frac{10 \text{ kN} \times 4 \text{ m}}{8} - \frac{10 \text{ kN} \times 4 \text{ m}}{2} = -25 \text{ kN} \cdot \text{m} \\ M_{CA}^F &= -\frac{3 \times 10 \text{ kN} \times 4 \text{ m}}{8} - \frac{10 \text{ kN} \times 4 \text{ m}}{2} = -35 \text{ kN} \cdot \text{m} \end{aligned}$$

For column EG, in addition to a load of this story, it is also subjected to the shear 20 kN at the column top, thus the fixed-end moments are calculated as

$$M_{EG}^F = -\frac{10 \text{ kN} \times 4 \text{ m}}{8} - \frac{20 \text{ kN} \times 4 \text{ m}}{2} = -45 \text{ kN} \cdot \text{m}$$

$$M_{GE}^F = -\frac{3 \times 10 \text{ kN} \times 4 \text{ m}}{8} - \frac{20 \text{ kN} \times 4 \text{ m}}{2} = -55 \text{ kN} \cdot \text{m}$$

The remaining calculation is shown in figure 6.45b, and diagram M is shown in figure 6.45c.

6.10 Characteristics of Statically Indeterminate Structures

Compared with statically determinate structures, statically indeterminate structures have the following important mechanical characteristics. Understanding these characteristics can help to deepen the understanding of statically indeterminate structures and make better use of them.

(1) The internal forces of a statically determinate structure can only be determined by the static equilibrium conditions, and these values are independent of the material properties of the structure and the cross-sectional dimension and stiffness ratio of the members. However, the internal forces of a statically indeterminate structure cannot be uniquely determined only by the static equilibrium conditions, and deformation coordination conditions must also be considered. Therefore, the internal forces of the statically indeterminate structure are related to the material properties of the structure and the cross-sectional dimension, and the stiffness ratio of the members.

Because of this characteristic, the cross-sectional dimension of each member or its relative value must be determined in advance before calculating the statically indeterminate structure. However, since the internal forces have not yet been calculated, it is usually based on experience or a simple method to approximate the cross-sectional dimensions of the members, which is the basis for calculation. Then, the required cross-section is selected according to the calculated internal forces, which of course does not necessarily coincide with the pre-determined cross-section. So, it requires re-adjusting the cross-section and then recalculating. This process is repeated until a satisfactory result is obtained. Therefore, the design process of a statically indeterminate structure is more complicated than that of a statically determinate structure. In addition, this characteristic can also be used to adjust the internal force distribution of the statically indeterminate structure by changing the stiffness ratio of each member to achieve the desired purpose.

(2) In a statically determinate structure, other factors (such as support movement, temperature changes, material shrinkage, manufacturing errors, etc.) will not cause internal force except loads. However, in a statically indeterminate structure, any of the above factors can usually cause additional internal forces. This is because the above factors will cause structural deformation. Due to the restriction of redundant constraints of the structure, such deformation often causes the internal forces of the structure, which are generally proportional to the absolute value of the stiffness of each member.

This characteristic of the statically indeterminate structure may bring adverse effects under certain conditions. For example, a continuous beam may generate excessive additional internal forces due to uneven subsidence of the foundation. However, it may become a beneficial aspect in other cases. For example, the internal forces of a continuous beam can be adjusted by changing the height of the supports to obtain a more reasonable distribution of internal forces.

(3) A statically determinate structure will lose its geometric stability when any constraint is broken, and then it can no longer bear loads. However, a statically indeterminate structure still maintains its geometric stability after the redundant constraints are broken, so it still has a certain bearing capacity. Therefore, from the perspective of military engineering and structural seismic resistance, statically indeterminate structures have strong disaster prevention capabilities.

(4) Due to the existence of redundant constraints, the influence of local loads on the internal force and deformation of statically indeterminate structures is larger than that of statically determinate structures. As shown in figure 6.46a, when the mid-span of the three-span continuous beam is loaded, the two side spans will also generate internal forces and deformation. However, the three-span simply supported beam shown in figure 6.46b is different. The two side spans do not generate internal forces and deformation when the mid-span is loaded, and the maximum deflection of the mid-span (figure 6.46d) is larger than that of the continuous beam (figure 6.46c). Moreover, the distribution of the internal force of statically indeterminate structures is more uniform than statically determinate structures, and the peak value of the internal force is also smaller. In addition, from the perspective of structural stability, the critical pressure of a beam fixed at one end and slider-supported at the other end is four times that of the beam pinned at both ends. Therefore, the stiffness and stability of statically indeterminate structures are improved due to the existence of redundant constraints.

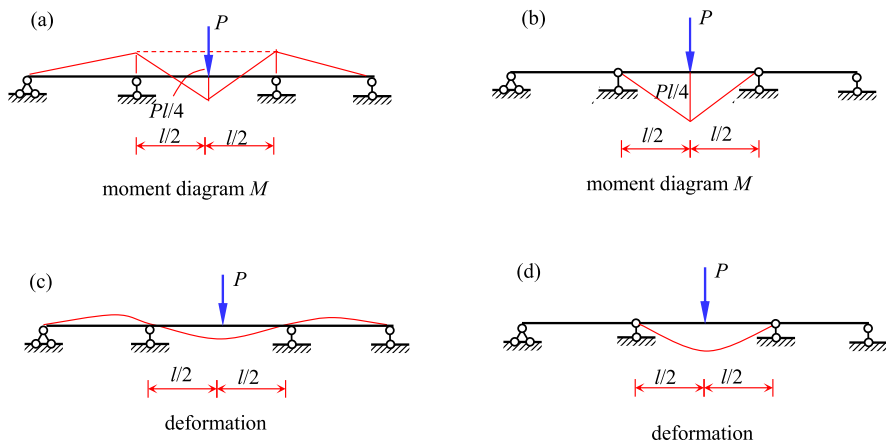


FIG. 6.46 – (a) Moment diagram of three-span continuous beam, (b) moment diagram of three-span simply supported beam, (c) deformation of three-span continuous beam, and (d) deformation of three-span simply supported beam.

Problems

6.1 Write the expressions of the member end moment and the basic equation of the displacement method.

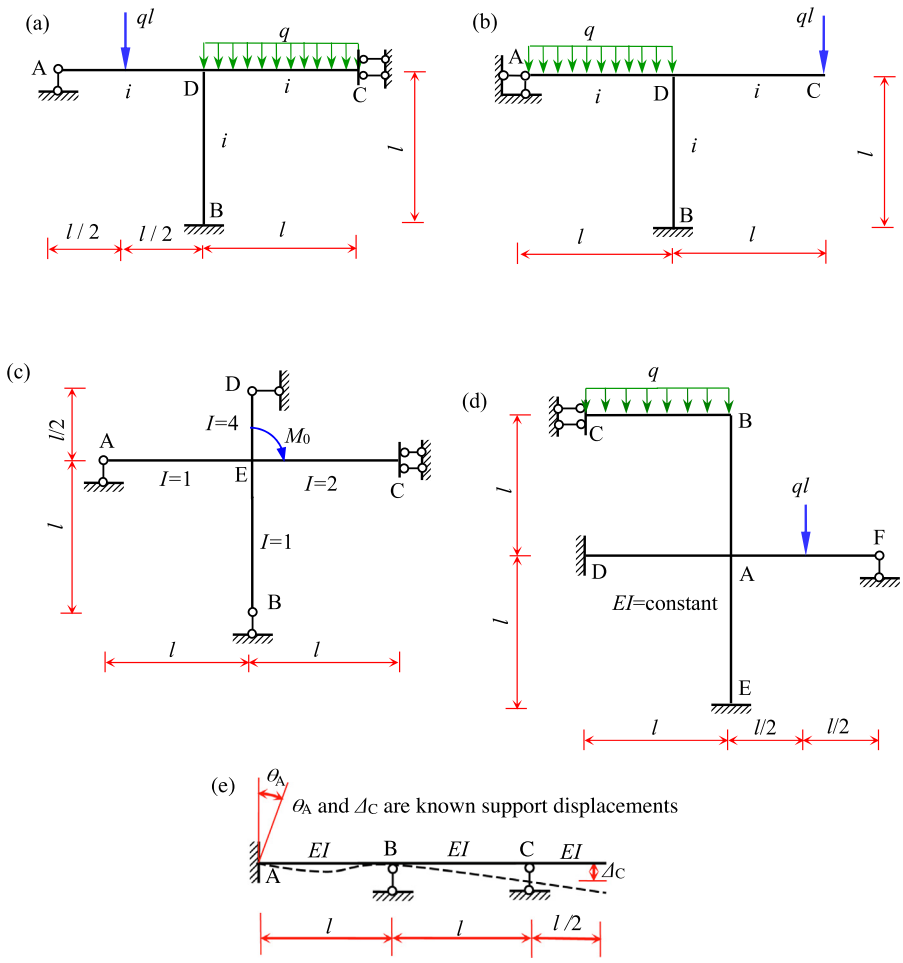


FIG. P6.1

6.2 Calculate the axial force of each member in the frame by displacement method. The EA is the same for each member.

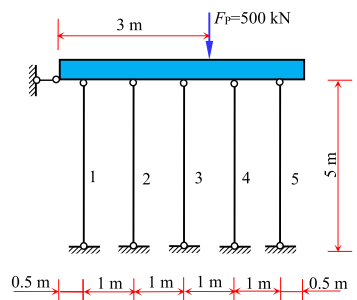


FIG. P6.2

6.3 Draw the moment diagram M of the frame. The EI is the same for each member.

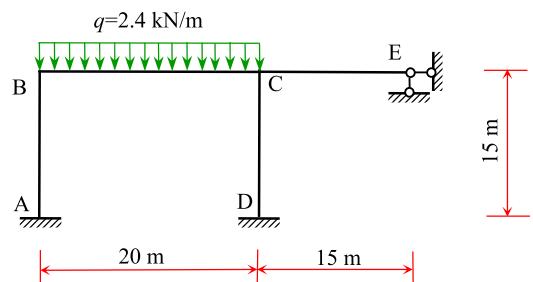


FIG. P6.3

6.4 Draw the diagram M of the bent shown in the figure.

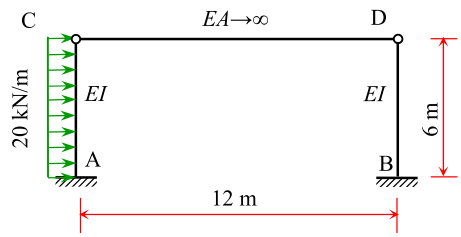


FIG. P6.4

6.5 Draw the diagram M of the frame shown in the figure.

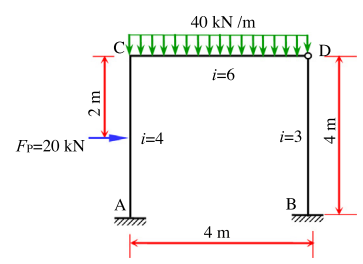


FIG. P6.5

6.6 Draw the internal force diagram of the frame.

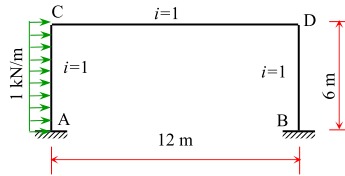


FIG. P6.6

6.7 Draw the moment diagram M of the symmetric frame by using the symmetry.

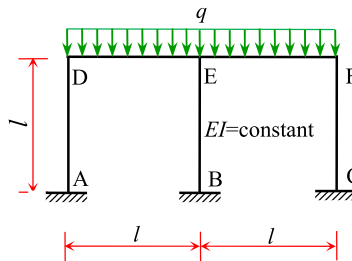


FIG. P6.7

6.8 Draw the moment diagram M of the symmetric frame by using the symmetry.

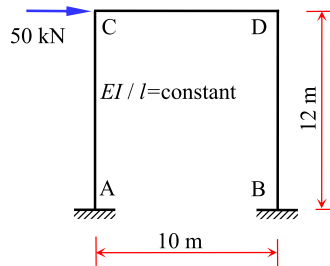


FIG. P6.8

6.9 Draw the diagram M of the symmetric frame by using the symmetry.

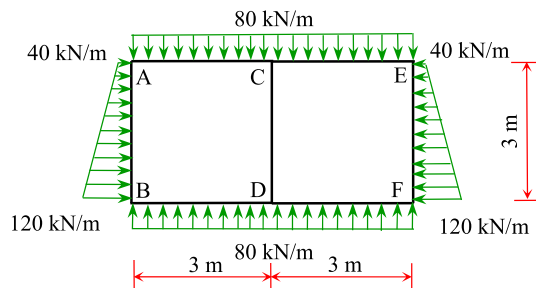


FIG. P6.9

6.10 Draw the moment diagram M of the frame. $l = 10$ m, E is constant, the uniformly distributed load is q .

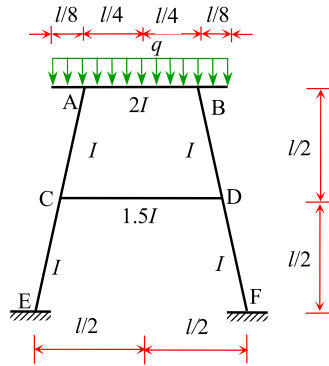


FIG. P6.10

6.11 Draw the moment diagram M of the symmetric frame by using the symmetry. The EI is the same for each member, $F_P = 20$ kN, $q = 10$ kN/m.

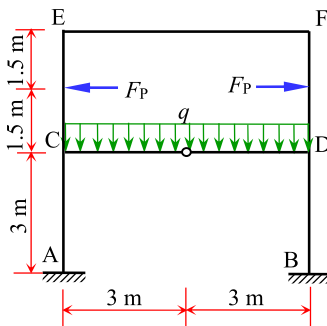


FIG. P6.11

6.12 Draw the moment diagram M of the frame, and EI is the same for each member.

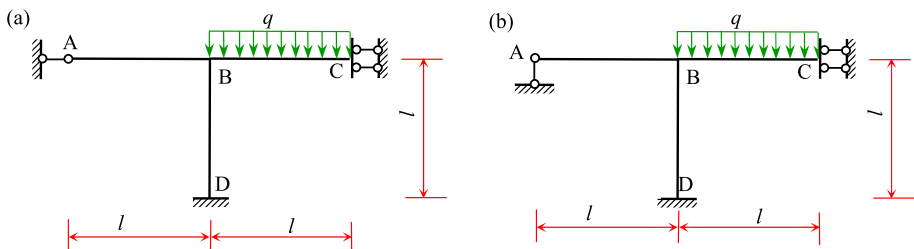


FIG. P6.12

6.13 Draw the moment diagram M of the frame, and EI is the same for each member.

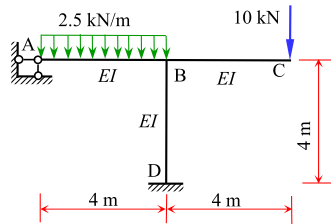


FIG. P6.13

6.14 Draw the moment diagram M of the frame, and EI is the same for each member.

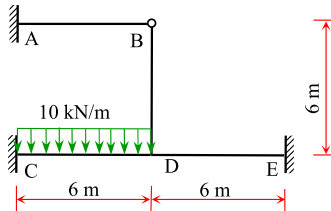


FIG. P6.14

6.15 Use the moment-distribution method to draw the moment diagram M of the beam.

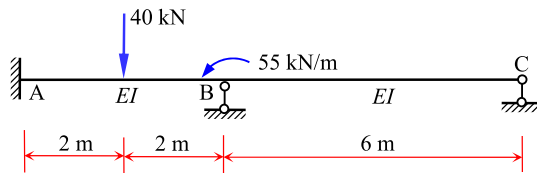


FIG. P6.15

6.16 Apply the no-shear distribution method to draw the moment diagram M of the frame.

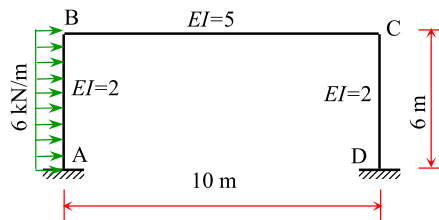


FIG. P6.16

6.17 Utilize the moment-distribution method to draw the moment diagram M of the frame.

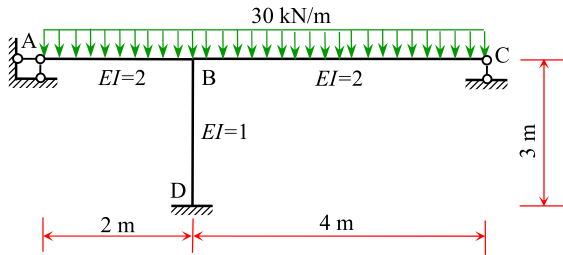


FIG. P6.17

6.18 Construct the moment diagram of the open web beam (also called the open web truss), and calculate the vertical displacement of joint F.

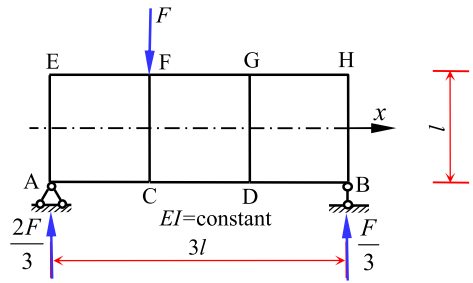


FIG. P6.18

Chapter 7

Influence Lines for Structures under Moving Loads

For the force analysis of structures discussed in the previous chapters, the loads considered are fixed loads whose magnitudes and directions are unchanged, and the loading positions are fixed. However, some structures are subjected to not only the fixed loads (such as the dead loads of structural weight or machine weight and so on) but also the moving loads whose positions change along the structures, for examples, the trains and automobiles driving on the bridges, the cranes traveling on the rails, as well as the distributed loads which can be placed arbitrarily such as the crowd, goods, or non-fixed equipment on the building floor. All of them are moving loads and also belong to the live loads.

This chapter will investigate how to construct the influence lines and calculate the internal forces of statically determinate and indeterminate structures under moving loads. There are two methods for constructing the influence lines of internal forces or the support reactions of structures, namely, the equilibrium method and the kinematic method, in which the kinematic method is simpler and more convenient. The concept of influence line was independently introduced by Otto Mohr and Emil Winkler. In 1886, Heinrich Muller-Breslau (1851 ~ 1925) proposed a method for rapidly determining the shape of influence line for the internal force of statically determinate and indeterminate beams based on the principle of virtual displacements. Namely, the influence line shape of a certain force is the same as the deflected shape of a beam, in which the corresponding restraint under this force is removed. This method is well known as the Muller-Breslau principle or the Muller-Breslau criterion, also named the kinematic method. In 2015, Dixiong Yang *et al.* utilized the principle of virtual displacements of the deformable body to establish a kinematic method for exactly, analytically, and conveniently calculating the influence line equations of internal forces of the statically indeterminate beam structures, and then rapidly constructing corresponding influence lines. At present, the influence lines have important applications in the analysis and design of bridge structures under moving loads as shown in figure 7.1 and structural health monitoring.



FIG. 7.1 – Bridge structure under moving loads.

7.1 Concepts of Moving Load and Influence Line

Generally, a moving load refers to the live load whose magnitude and acting direction remain unchanged, but whose position moves along the structure. The internal force calculation of the structure subjected to a moving load has the following characteristic: the load is still static, while the internal force of the structure varies with the movement of the load. Thus, it is necessary to study the change range and regulation of internal force under moving load. In fact, the possible maximum value of internal force must be taken as the design basis. Therefore, the most unfavorable position of loads, that is, the loading position where a certain internal force or reaction reaches the maximum should be determined.

There are many types of moving loads. It is not necessary to discuss each one of them, but it is suitable only to extract the load characteristic that is common for typical analysis. First, a typical case is selected from the general. The typical case is the unit moving load $F_P = 1$, which is the simplest and most fundamental element extracted from various moving loads, and its dimension is 1 which means the unit moving load is dimensionless. Then, a general problem can be resolved through the typical case. As long as the varying regulation of internal forces of the structure under the unit moving load (namely, typical load) is analyzed clearly, according to the superposition principle, the internal forces under the actual moving loads and the most unfavorable position of the loads can be determined smoothly.

The diagram showing the change regulation of the internal force of the structure subjected to unit moving load is called the **influence line of internal force**. **Influence line** is a basic tool for studying the influence of moving load on the reaction, shear, and moment of the structure. The concept of the influence line will be illustrated through the following example.

The simply supported beam AB is shown in figure 7.2a. When a single vertical load F_P moves on the beam, the varying regulation of support reaction F_{RB} is discussed herein.

Point A is the origin of the coordinate system, and x represents the abscissa of the acting point of load F_P . If x is a constant, F_P is a fixed load. Conversely, if x is treated as a variable, F_P is the moving load. When F_P acts on any position

x ($0 \leq x \leq l$) over the beam, the support reaction F_{RB} can be obtained by the following equilibrium equation

$$F_{RB} = \frac{x}{l} F_P \quad (0 \leq x \leq l) \quad (a)$$

It is noted that F_{RB} is proportional to F_P , and the proportionality coefficient x/l is named as the **influence coefficient** of F_{RB} and can be expressed by $\bar{F}_{RB}$

$$\bar{F}_{RB} = \frac{x}{l} \quad (0 \leq x \leq l) \quad (b)$$

Obviously, the influence coefficient $\bar{F}_{RB}$ is numerically equal to the reaction F_{RB} caused by unit moving load $F_P = 1$, and its dimension is also 1.

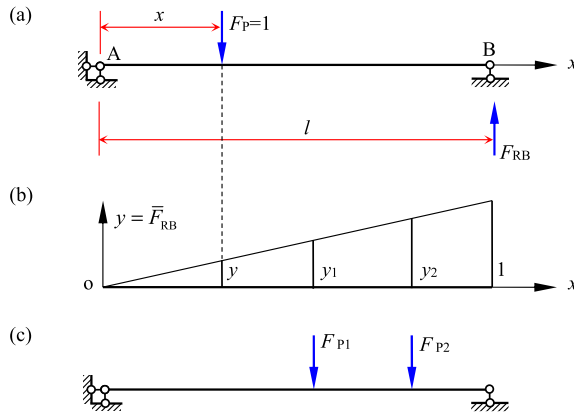


FIG. 7.2 – (a) Simply supported beam under unit moving load, (b) influence line for reaction F_{RB} , (c) wheel pressures F_{P1} and F_{P2} of crane.

Equation (b) represents the functional relationship between the influence coefficient $\bar{F}_{RB}$ and loading position parameter x , the graph of this function is called the influence line for F_{RB} . Since the right-hand term of equation (b) is a linear function of x , the influence line for F_{RB} is a straight line. To determine the two points on the line, we can suppose $x = 0$ to achieve $\bar{F}_{RB} = 0$, and assume $x = l$ to get $\bar{F}_{RB} = 1$. By connecting these two points on a straight line, the influence line for F_{RB} is obtained, as shown in figure 7.2b.

The influence line in figure 7.2b demonstrates the changing law of support reaction F_{RB} with $F_P = 1$: when $F_P = 1$ gradually moves from point A to point B, the influence coefficient of reaction $\bar{F}_{RB}$ starts from zero, gradually increases, and finally reaches the maximum $\bar{F}_{RB} = 1$.

The influence line for F_{RB} can also be employed to determine the reaction F_{RB} caused by various loads. For instance, the beam shown in figure 7.2c has the wheel pressures F_{P1} and F_{P2} of crane. According to the superposition principle, the support reaction F_{RB} can be calculated by

$$F_{RB} = F_{P1} y_1 + F_{P2} y_2$$

where, y_1 and y_2 indicate the influence coefficients $\bar{F}_{RB1}$ and $\bar{F}_{RB2}$ corresponding to the positions of loads F_{P1} and F_{P2} , respectively.

In summary, when the unit concentrated load $F_P = 1$ moves vertically along the structure, the curve representing the changing law of the structural response Z is just the influence line for Z . The abscissa x of any point on the influence line means the position parameter of the load, and the ordinate y represents the influence coefficient $\bar{Z}$ of response Z in the structure under the load applied on this point.

The influence coefficient $\bar{Z}$ is a proportional coefficient between the structural response Z and load F_P , namely

$$Z = F_P \bar{Z}, \quad \text{or} \quad \bar{Z} = \frac{Z}{F_P}$$

When $F_P = 1$, $\bar{Z}$ is numerically equal to Z . However, the difference between $\bar{Z}$ and Z is the dimension of load F_P . If the dimension of F_{RB} is LMT^{-2} , then the dimension of $\bar{F}_{RB}$ is $\text{LMT}^{-2}/\text{LMT}^{-2}$, which is a quantity of dimension 1.

7.2 Equilibrium Method for Constructing Influence Lines of Simply Supported Beams

The **equilibrium method** can be applied to construct the influence lines for the internal forces or reactions of the statically determinate structures, in which the variable is the loading position x , and the static equilibrium equations are established to determine the influence line equations of the internal forces (or reactions), and then the influence lines are drawn. Hereafter, the procedure of the equilibrium method is described by taking the influence lines for internal forces (or reactions) of the simply supported beam as examples.

1. Influence line for support reaction

The influence line for the reaction F_{RB} for a simply supported beam (figure 7.3b) has been discussed in section 7.1 (figure 7.2). The influence line for the support reaction F_{RA} will be discussed below.

$F_P = 1$ moves on the arbitrary position of the beam, and the distance from support A is x . Calculate the influence coefficient $\bar{F}_{RA} = F_{RA}/F_P$ in terms of the equilibrium equation:

$$\sum M_B = 0, \quad \bar{F}_{RA}l - 1(l - x) = 0$$

then

$$\bar{F}_{RA} = \frac{l - x}{l} \quad (0 \leq x \leq l)$$

This is the influence line equation of $\bar{F}_{RA}$. It can be seen that the influence line for F_{RA} is also a straight line. At point A, $x = 0$, and $\bar{F}_{RA} = 1$. At point B, $x = l$, and $\bar{F}_{RA} = 0$. Using these two ordinates to connect a straight line, we can draw the influence line for F_{RA} in figure 7.3c. The upward support reaction is regarded as positive value, and the ordinates of influence line with positive values are drawn above the baseline (*i.e.*, axis of beam).

2. Influence line for shear

Now construct the influence line for shear F_{SC} at the specified section C. When $F_P = 1$ is applied at the left or right side of point C, the influence coefficient of F_{SC} has different expressions, which should be considered separately. When $F_P = 1$ is applied on segment CB, the segment AC is regarded as a free body. By using $\sum F_y = 0$, we have

$$F_{SC} = F_{RA} \quad (F_P = 1 \text{ in segment CB})$$

It is seen that in segment CB, the influence line for F_{SC} is the same as that of F_{RA} . Thus, the influence line for F_{RA} can be drawn first, and then the diagram of segment CB remains (the diagram of segment AC is discarded). The ordinate of point C can be determined as b/l according to the proportional relation.

When $F_P = 1$ moves on segment AC, segment CB is taken as a free body. Based on $\sum F_y = 0$, we have

$$F_{SC} = -F_{RB} \quad (F_P = 1 \text{ in segment AC})$$

In segment AC, the influence line for F_{SC} is the same as that of F_{RB} , but its sign is opposite. Therefore, the influence line for F_{RB} is turned over and drawn below the baseline, and then the diagram of segment AC remains. The ordinate of point C can be calculated as $-a/l$ according to the geometrical relation.

To sum up, the influence line for shear F_{SC} is divided into segments AC and CB, which consist of two parallel lines, and forms a step at point C as shown in figure 7.3d. When $F_P = 1$ is applied on any point of segment AC, the shear at section C is negative. When $F_P = 1$ moves on any point of segment CB, the shear at section C is positive. When $F_P = 1$ moves through point C from the left to right side, the shear of section C will change abruptly with a value of 1. When $F_P = 1$ is just applied at point C, the influence coefficient of F_{SC} is meaningless.

The influence coefficient of shear $\bar{F}_S = F_S/F_P$ is a proportional coefficient, which is a quantity with the dimension 1.

3. Influence line for bending moment

Now draw the influence line for bending moment M_C of the specified section C. Two cases are still considered, namely, $F_P = 1$ is applied at the left or right side of section C.

When $F_P = 1$ is applied on segment CB, the segment AC is taken as a free body, and

$$M_C = F_{RA} \cdot a \quad (F_P = 1 \text{ on segment CB})$$

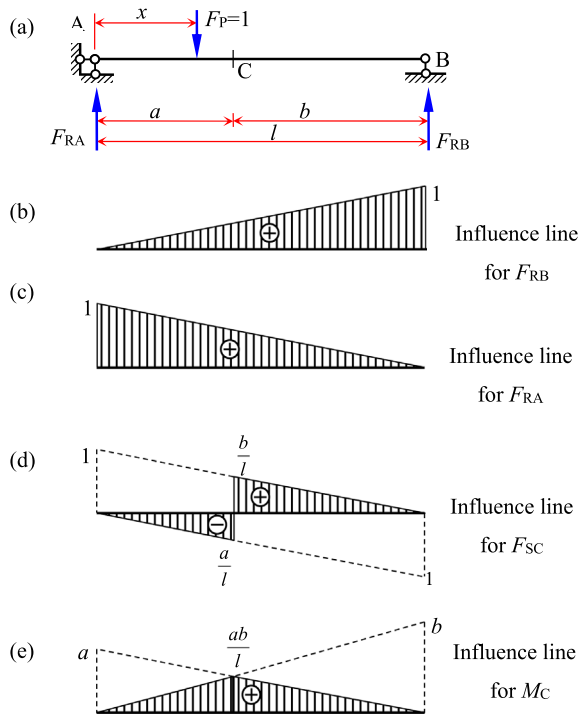


FIG. 7.3 – Influence lines for reactions and internal forces of simply supported beam.

In segment CB, the influence coefficient of M_C is equal to a times of that of F_{RA} . Hence, we can multiply the ordinate of the F_{RA} 's influence line by a , and then keep it in segment CB to achieve the M_C 's influence line in segment CB. Note that the ordinate of point C should be ab/l .

When $F_P = 1$ is applied on segment AC, the segment CB is taken as a free body. We have

$$M_C = F_{RB} \cdot b \quad (F_P = 1 \text{ on segment AC})$$

Therefore, the ordinate of the influence line for F_{RB} is multiplied by b , and the diagram of segment AC is remained, and then the influence line for M_C in segment AC can be drawn. Herein, the ordinate of point C is still ab/l .

In summary, the influence line for M_C is divided into two segments AC and CB. Each segment is a straight line, and they form a triangle, as shown in figure 7.3e. When the unit load acts on point C, $\overline{M}_C$ is the largest. When $F_P = 1$ moves from point C to both ends of the beam, $\overline{M}_C$ gradually decreases to zero.

The influence coefficient $\overline{M} = M/F_P$ of bending moment is a proportional one, whose dimension is L and the unit is m.

Finally, take the simply supported beam in figure 7.4 as an example. Table 7.1 lists the differences between the influence line and internal force diagram of the

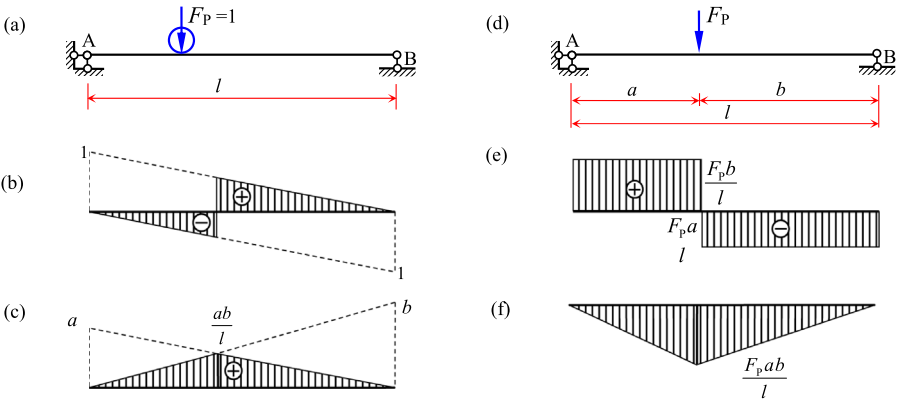


FIG. 7.4 – Influence lines and internal force diagrams of simply supported beam.

TAB. 7.1 – Differences between influence line and internal force diagram of structure.

	Influence line	Internal force diagram
Load magnitude	$F_P = 1$	Actual F_P
Load property	Moving	Fixed
Abscissa	Indicates the load position	Indicates the position of section
Ordinate	Represents the change of internal force in a section	Represents the distribution of internal forces in all sections
Graphic range	The segments where $F_P = 1$ moves	Entire structure
General rules for drawing	The positive value is drawn above the baseline, and the sign is marked	M diagram is drawn on the tensile side without any sign. The sign rules of diagrams F_S and F_N are the same as those of influence line for internal force
Dimension of internal force	M : L, F_S , F_N : dimension 1	M : L^2MT^{-2} , F_S , F_N : LMT^{-2}

structure. Figure 7.4b, c indicate the influence lines for the shear and bending moment of a simply supported beam, respectively. Figure 7.4e, f illustrate the shear and bending moment diagrams of the simply supported beam in figure 7.4d, separately.

Example 7.1. Draw the influence lines for F_{RA} , F_{RB} , F_{SC} , and F_{SD} of the cantilever beam as shown in figure 7.5a.

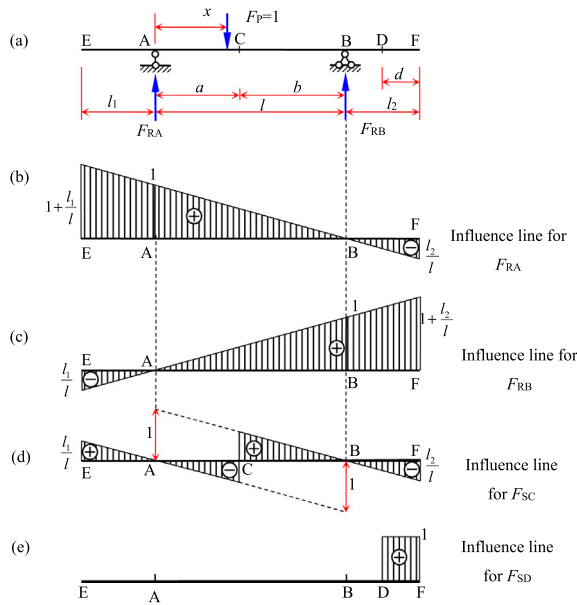


FIG. 7.5 – Influence lines for reactions and internal forces of cantilever beam.

Solution: (1) Draw the influence lines for F_{RA} and F_{RB}

Point A is the origin of the coordinate, and the abscissa x is positive when it points to the right. When the unit moving load $F_P = 1$ is applied at any position x on the beam, the influence coefficients of the support reactions are achieved by using the equilibrium equation as follows

$$\left. \begin{aligned} \bar{F}_{RA} &= \frac{l-x}{l} \\ \bar{F}_{RB} &= \frac{x}{l} \end{aligned} \right\} \quad (-l_1 \leq x \leq l + l_2)$$

The influence line equations of the two support reactions are identical to those of the simply supported beam, except that the scope of action of $F_P = 1$ is extended. For a simply supported beam, the change range of x is $0 \leq x \leq l$, and for this cantilever beam, the variation range is $-l_1 \leq x \leq l + l_2$. The influence line diagrams are shown in figure 7.5b, c. The influence line within segment AB is identical to that of the simply supported beam, and it is still a straight line. Subsequently, the line is extended to the two cantilever portions to obtain the entire influence line.

(2) Draw the influence line for shear F_{SC}

When $F_P = 1$ is applied on the left part of point C, we have

$$F_{SC} = -F_{RB} \quad (F_P = 1 \text{ in the EC segment})$$

When $F_P = 1$ is applied on the right part of point C, then

$$F_{SC} = F_{RA} \quad (F_P = 1 \text{ in the CF segment})$$

The influence line for F_{SC} is obtained by using the method similar to that of simply supported beams, as shown in figure 7.5d.

(3) Draw the influence line for shear F_{SD}

When $F_P = 1$ acts on the left part of point D, the right part of D is taken as a free body, we have

$$\overline{F}_{SD} = 0$$

When $F_P = 1$ acts on the right part of point D, the right part of D is still taken as a free body, thus

$$\overline{F}_{SD} = 1$$

The influence line for F_{SD} can be drawn as shown in figure 7.5e. Only the influence coefficient of segment DF is not zero. In other words, the shear F_{SD} is affected only when the load acts on segment DF.

7.3 Influence Lines for Girders and Trusses

7.3.1 Influence Lines for Internal Forces of Girders

Figure 7.6a indicates a schematic diagram of the floor beam and girder system in the bridge structure. The floor loads are applied on the stringers. The stringers are regarded as simply supported beams with two ends resting on the floor beams. The floor beams are supported by the girder, and the floor loads are transmitted from the stringers to the floor beams, then to the side girders. Regardless of the loads on the stringers, the girders are only subjected to concentrated forces at the locations of the floor beams such as joints A, C, E, F, and B, etc., hence the girders are under the **joint loads**. There are two types of loading for beam bridge: direct load (such as the load acting on the stringers) and joint load (also called indirect load, such as the load exerting on the girders).

The method of drawing the influence lines for support reaction and internal force of the girder under joint loads are illustrated as follows.

(1) Draw the influence lines for support reaction F_{RA} and F_{RB} .

The influence lines for support reactions F_{RA} and F_{RB} are exactly the same as those shown in figure 7.3c, b, which are not drawn in figure 7.6.

(2) Draw the influence line for M_C .

Point C is the key point. When $F_P = 1$ is applied on the right side of point C, M_C is obtained by F_{RA} , and when $F_P = 1$ is applied on the left side of point C, M_C is

obtained by F_{RB} . The influence line for M_C is exactly the same as that of figure 7.3e, as shown in figure 7.6b. The ordinate of point C is

$$\frac{ab}{l} = \frac{d \times 3d}{4d} = \frac{3}{4}d$$

(3) Draw the influence line for M_D .

The influence line for M_D is shown in figure 7.6c.

First, assume that $F_P = 1$ is applied on the girder AB, then the influence line for M_D is a triangle (where the CE segment is a dashed line). The ordinate of point D is

$$\frac{ab}{l} = \frac{\frac{3}{2}d \times \frac{5}{2}d}{4d} = \frac{15}{16}d$$

In terms of the proportional relation, the ordinates of points C and E are

$$y_C = \frac{15}{16}d \times \frac{2}{3} = \frac{5}{8}d \quad y_E = \frac{15}{16}d \times \frac{4}{5} = \frac{3}{4}d$$

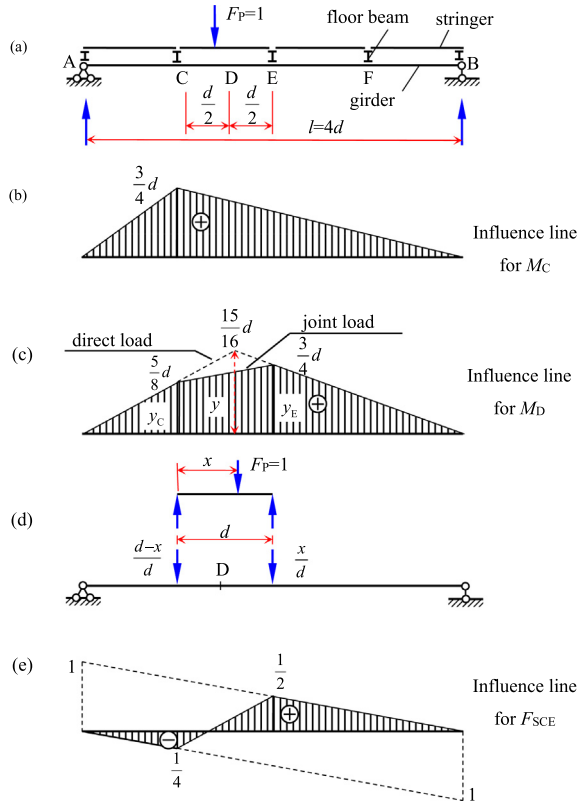


FIG. 7.6 – Influence lines for internal forces of girder under joint loads.

By connecting the ordinates of the two points C and E by a straight line, the influence line of M_D under the joint load is obtained, as shown by the solid line in the figure. In order to prove the correctness of the above approach, only the following two points need to be paid attention:

- 1) If the unit load is applied on point C or point E, the joint load is exactly the same as the direct load. Therefore, under the joint load, the ordinates y_C and y_E of the influence line of M_D at points C and E are equal to the corresponding ordinate under direct load.
- 2) If the unit load acts between the two points C and E of the stringer, and the distance to point C is expressed by x , the reaction of the stringer CE is shown in figure 7.6d. The girder bears a downward load $(d - x)/d$ at point C and a downward load x/d at point E. The influence coefficient of moment M_D of the girder when unit load acting at the point x can be obtained by the superposition principle

When $F_P = 1$ is applied at point C, $\overline{M}_D = y_C$

When $F_P = 1$ is applied at point E, $\overline{M}_D = y_E$

When the distance between $F_P = 1$ and point C is x , the load at point C of the girder is $\frac{d-x}{d}$, and the load at point E is $\frac{x}{d}$, so

$$y = y_C \frac{d-x}{d} + y_E \frac{x}{d}$$

The above equation is an expression of the linear function. It is seen that under the joint loads, the influence line for M_D is a straight line in the panel CE.

The general conclusions can be expressed as follows:

- 1) Under the joint loads, any influence line of structure is a straight line between two adjacent joints.
- 2) The influence line under the direct load is constructed first, and then the ordinates of the two adjacent joints are connected by a straight line. In this way, the **influence line of the structure under joint load** is obtained.
- 3) Draw the influence line for F_{SCE} .

Under the joint loads, the girder has no external force between the two joints C and E, so the shear of each section of panel CE is equal, which is usually called the panel shear, and is expressed by F_{SCE} . The influence line for F_{SCE} is shown in figure 7.6e, which is drawn based on the above conclusions.

7.3.2 Influence Lines for Axial Forces of Trusses

Figure 7.7a shows a parallel chord truss in bridge engineering. Suppose the unit load moves along the bottom chord AG of the truss, and constructs the influence lines for

axial forces of members. The truss is usually subjected to joint loads, and the load transference method is the same as the girder system with stringers, floor beams, and girders as shown in figure 7.7b. The problem of constructing the influence line for the axial force of the truss using the equilibrium method can be regarded as the issue of drawing the force influence line of the truss under joint load. The influence line for the axial force of any bar (*e.g.*, F_{Nbc}) is a straight line between adjacent joints. The unit load $F_P = 1$ can be placed on the points of A, B, C, D, E, F, and G in turn, and the value of F_{Nbc} is calculated and expressed by the ordinates, and then connected with straight lines to obtain the influence line for F_{Nbc} . It is seen that in principle, there is no difficulty in constructing the influence line for internal force of the member of a statically determinate truss, but this way is fairly cumbersome. The equilibrium method for constructing the influence lines for axial forces of truss is described as follows *via* combining the example in figure 7.7, and its basis is still the method of sections and the method of joints, which is more convenient.

1. Influence lines for support reaction F_{RA} and F_{RG}

The influence lines for F_{RA} and F_{RG} are the same as those of simply supported beams, which are omitted herein.

2. Influence line for axial force F_{Nbc} of top chord bc

Set the section I-I, take joint C as the moment center, and use the torque equation $\sum M_C = 0$ to calculate F_{Nbc} .

If the unit load is applied on the right side of joint C, take the left portion of section I-I as a free body, thus

$$F_{RA} \times 2d + F_{Nbc} h = 0$$

$$F_{Nbc} = -\frac{2d}{h} F_{RA} \quad (a)$$

If the unit load is applied on the left side of joint C, take the right portion of section I-I as free body, we achieve

$$F_{RG} \times 4d + F_{Nbc} h = 0$$

$$F_{Nbc} = -\frac{4d}{h} F_{RG} \quad (b)$$

In figure 7.7c, in terms of equation (a) the influence line for F_{RA} is drawn. Its ordinates are multiplied by $\frac{2d}{h}$, the influence line is drawn below the baseline, and then the right segment of joint C is retained. Similarly, according to equation (b) the influence line for F_{RG} is constructed. Its ordinates are multiplied by $\frac{4d}{h}$, the influence line is drawn below the baseline, and then the left segment of C retained. In this way a triangle is gotten. Since the diagrams between adjacent joints are straight lines, the obtained triangle is the influence line for F_{Nbc} .

Moreover, equation (a) and equation (b) can be merged into one formula, namely:

$$F_{\text{Nbc}} = -\frac{M_{\text{C}}^0}{h} \quad (\text{c})$$

where M_{C}^0 is the bending moment of joint C of the corresponding simply supported beam as figure 7.7b. It is known from equation (c) that, the influence line for F_{Nbc} is a triangle, and the ordinate of the vertex is:

$$-\frac{ab}{lh} = -\frac{2d \times 4d}{6dh} = -\frac{4d}{3h}$$

3. Influence line for axial force F_{NCD} of bottom chord CD

The section II-II is set, and the joint c is taken as the moment center to establish the moment equation $\sum M_{\text{C}} = 0$. Thus,

$$F_{\text{NCD}} = \frac{M_{\text{C}}^0}{h} \quad (\text{d})$$

The moment influence line of joint C of the corresponding beam is drawn, and then its ordinates are divided by h to obtain the influence line for F_{NCD} as shown in figure 7.7d.

4. Influence line for vertical component $F_{y\text{bC}}$ of axial force of diagonal bar bC

Still using section I-I, the influence line is considered in three segments.

When the unit load is applied on the right part of joint C, we consider the equilibrium condition of the left part of section I-I. According to the projection equation $\sum F_y = 0$, we have

$$F_{y\text{bC}} = F_{\text{RA}} \quad (\text{e})$$

When the unit load is applied on the right of joint B, the equilibrium condition of the right part of section I-I is expressed by

$$F_{y\text{bC}} = -F_{\text{RG}} \quad (\text{f})$$

If the unit load is moved between joint B and joint C, the influence line is a straight line.

The influence line for $F_{y\text{bC}}$ based on the above analysis is displayed in figure 7.7e. The shear F_{SBC}^0 of panel BC of the corresponding beam is utilized, and the above analysis can be summarized into a formula:

$$F_{y\text{bC}} = F_{\text{SBC}}^0 \quad (\text{g})$$

The influence line exhibited in figure 7.7e is actually that of panel shear F_{SBC}^0 of the corresponding beam.

5. Influence line for axial force F_{NcC} of vertical bar cC

The section II-II is taken, and the projection equation $\sum F_y = 0$ is used to solve F_{NcC} . The shear F_{SCD}^0 of panel CD of the corresponding beam can be utilized to get the following equation:

$$F_{NcC} = -F_{SCD}^0 \tag{h}$$

Figure 7.7f illustrates the influence line for F_{NcC} , which is constructed according to the influence line for panel shear F_{SCD}^0 , but the sign is changed.

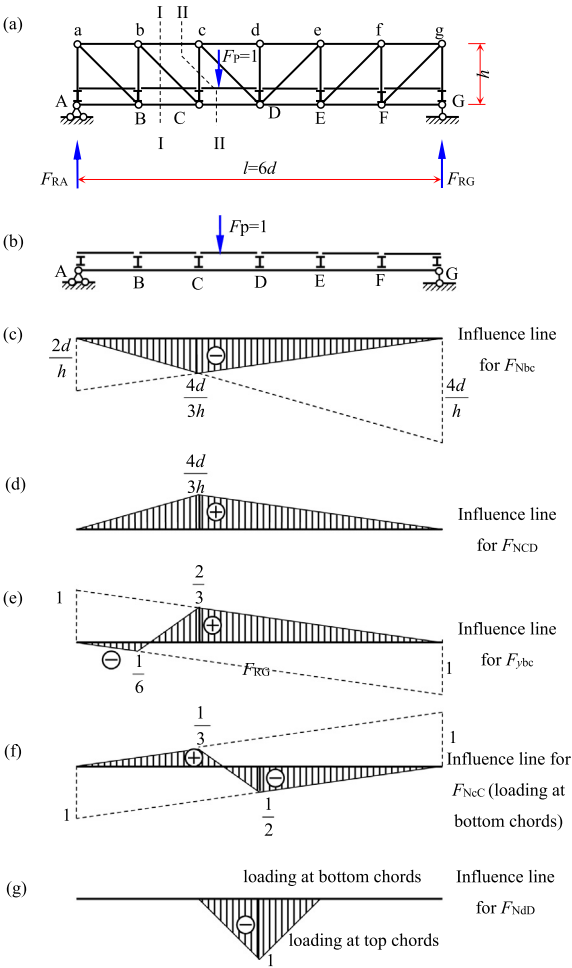


FIG. 7.7 – Influence lines for axial forces of plane truss.

6. Influence line for axial force F_{NdD} of vertical bar dD

In the above analysis, it is assumed that the unit load moves along the bottom chords. Namely, the load is carried by the joints of bottom chords of the truss (loading at bottom chords). The equilibrium condition of joint d of the top chord leads to

$$F_{\text{NdD}} = 0$$

Therefore, the influence line for F_{NdD} overlaps the baseline (figure 7.7g), and bar dD is always zero-bar regardless of the position of the unit moving load.

Assume that the unit load moves along the top chords of the truss, and the load is carried by the joints of top chords (loading at top chords). Considering the equilibrium of the joint d, we have:

$$\text{When } F_P = 1 \text{ is applied on joint d, } \bar{F}_{\text{NdD}} = -1$$

$$\text{When } F_P = 1 \text{ is applied on another joint, } \bar{F}_{\text{NdD}} = 0$$

The influence line for F_{NdD} is a triangle as shown in figure 7.7g.

When constructing the force influence line for the truss, it is necessary to distinguish whether the load is placed at bottom chords or top chords. In this example, if the load is changed to place at the top chords, the influence lines for F_{NbC} , F_{NcD} , and F_{ybC} are still shown in figure 7.7c–e. However, the influence line for F_{NcC} in figure 7.7f needs to be modified, because for the loading at top chords of truss, equation (h) is replaced by the following formula:

$$F_{\text{NcC}} = -F_{\text{SBC}}^0$$

The influence line for F_{NcC} should be drawn according to that of panel shear F_{SBC}^0 , but the sign is opposite (influence line for F_{NcC} is omitted for the loading at top chords of truss).

7.4 Kinematic Method for Constructing Influence Lines of Statically Determinate Structures

In addition to the equilibrium method, the kinematic method can also be used to solve the influence lines for the internal force or support reaction of statically determinate structures. The **kinematic method** is based on the principle of virtual work and can transform the static problem of the statically determinate force or support reaction into the geometric problem of a virtual displacement diagram. In fact, the basis of the kinematic method for influence line for statically determinate force is the principle of virtual work of rigid body system, and the kinematic method of influence line for statically indeterminate force introduced in section 7.6 is based on the principle of virtual work of the deformable body. The influence line equations derived from these two cases are the same, while the calculation processes are different.

The influence line for the support reaction of a simply supported beam is taken as an example as follows, to illustrate the concept and procedure for constructing the force influence lines of statically determinate structures by the kinematic method.

In order to draw the influence line for reaction $Z = F_{RB}$ of support B in the beam shown in figure 7.8a, the constraint corresponding to Z (roller support B) is removed, and replaced with the unknown force Z (figure 7.8b) to make the system have one degree of freedom and become a mechanism. Then, the equation of virtual work of the rigid body system is obtained by imposing the virtual displacements of the system and making the beam rotate slightly around point A:

$$Z\delta_Z + F_P\delta_P = 0 \quad (7.1)$$

where δ_P means the displacement corresponding to the load $F_P = 1$. Since the downward F_P is considered positive, the downward δ_P is also positive. δ_Z is the displacement corresponding to the unknown force Z , which is positive when it is consistent with the positive direction of Z . According to equation (7.1), we have

$$\bar{Z} = -\frac{\delta_P}{\delta_Z} \quad (7.2)$$

When $F_P = 1$ moves, the displacement δ_P changes as a function of the loading position x (figure 7.8b), while the displacement δ_Z is independent of x and is a constant. The influence coefficient $\bar{Z}(x)$ indicates the ratio of the two displacements in the rigid body system. For the sake of convenience, let $\delta_Z = 1$, that is, the unit displacement is generated along the positive direction of Z , and thus the influence line equation (7.2) is expressed as

$$\bar{Z}(x) = -\delta_P(x) \quad (7.3)$$

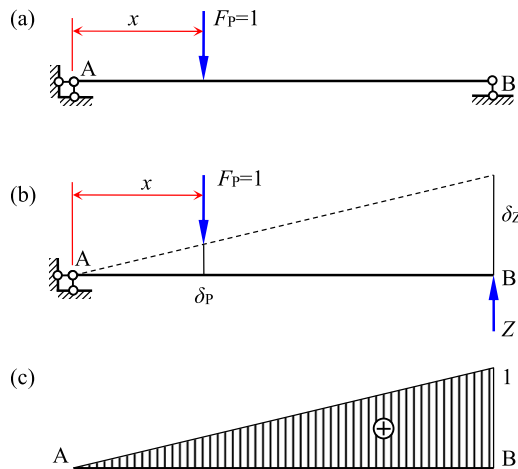


FIG. 7.8 – (a) Simply supported beam, (b) virtual displacements, (c) influence line for reaction F_{RB} .

The function $\bar{Z}(x)$ represents the corresponding ordinate of the influence line for response Z , and the function $\delta_P(x)$ indicates the corresponding ordinate of the vertical displacement diagram of the loading point when $\delta_Z = 1$ (figure 7.8c). According to equation (7.3), the virtual displacement δ_P diagram can be used to construct the influence line.

First, determine the shape of the influence line. Because the ratio of function $\bar{Z}(x)$ to function $\delta_P(x)$ is a constant, the influence line for Z has the same shape as the displacement δ_P diagram.

Subsequently, determine the ordinate values of the influence line. It is known from equation (7.3) that the ordinate $\bar{Z}(x)$ of influence line is equal to the corresponding ordinate in the displacement diagram of the rigid body system for $\delta_Z = 1$. The obtained figure 7.8c completely determines the influence line for Z from the aspects of shape and value.

Finally, the sign convention of ordinate of force influence line is determined. When δ_Z is a positive value, it is seen from equation (7.3) that the sign of Z is opposite to that of δ_P , and the downward δ_P is positive. Therefore, if the virtual displacement diagram is above the abscissa axis, the value of δ_P is negative, then the influence coefficient is positive.

To sum up, the procedure to determine the influence line for reaction or internal force of statically determinate structure *via* the kinematic method is presented as follows:

- (1) Remove the constraint corresponding to the response Z and apply the unknown force Z .
- (2) Let the rigid body system yield tiny virtual displacement δ_Z in the positive direction of Z , and the vertical displacement diagram (δ_P diagram) of the loading point is obtained. Thus, the shape of the influence line for Z can be rapidly determined.
- (3) Let $\delta_Z = 1$, and further determine the ordinate values of the influence line.
- (4) For the graph above the abscissa, the influence coefficient is taken as a plus sign, while for the graph below the abscissa, the influence coefficient is taken as a minus sign.

The advantage of the kinematic method lies in that it can quickly construct the shape of the influence line for internal force or support reaction without calculation, which is very helpful for engineering design. For example, when determining the most unfavorable position of loads, it is desirable only to know the shape of the influence line without having to calculate its value. In addition, the influence line addressed by the equilibrium method can also be checked by using the kinematic method.

Example 7.2. Utilize the kinematic method to construct the influence lines for bending moment and shear for the simply supported beam shown in figure 7.9a.

Solution: (1) Construct the influence line for bending moment M_C .

The constraint corresponding to bending moment M_C is removed, that is, section C is changed as a hinged joint, and a pair of couples M_C with equal values and

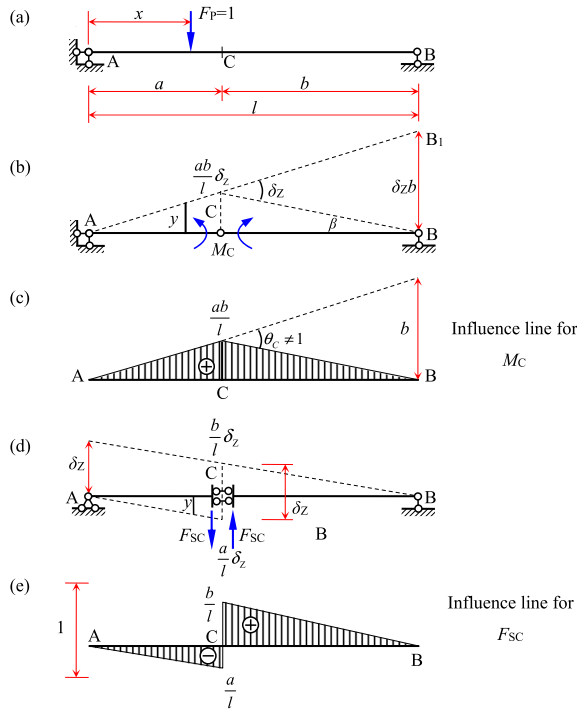


FIG. 7.9 – Influence lines for moment and shear force of simply supported beam by kinematic method.

opposite directions are applied. At this time, the rigid bodies at both sides of hinge C can rotate relatively.

Impose the system virtual displacements as shown in figure 7.9b. Here, the displacement δ_Z ($\delta_Z = \alpha + \beta$) corresponding to M_C is the relative rotation of two sides of hinge C. The ordinate in the displacement diagram can be determined by using δ_Z . Since δ_Z is a tiny rotation, $BB_1 = \delta_Z b$ is obtained first. According to the geometric relation, the vertical displacement of point C is $(ab/l)\delta_Z$. Thus, the resultant displacement diagram represents the shape of the influence line for M_C .

To obtain the values of influence coefficients, the displacement diagram in figure 7.9b is transformed as follows: the ordinate is enlarged, and δ_Z is set to 1, then the influence line for M_C is drawn in figure 7.9c, where the influence coefficient of point C is ab/l .

It should be pointed out that when performing the enlargement and transformation of the virtual displacement diagram, only the ordinate is enlarged ($1/\delta_Z$) times. That is, the parameter δ_Z in the ordinate is changed to 1, instead of changing the relative rotation at the hinge C to 1 rad.

(2) Construct the influence line for shear F_{SC} .

The constraint corresponding to the shear at section C is removed, and the shear F_{SC} is applied. The mechanism shown in figure 7.9d is obtained. At this time, a relative vertical displacement can occur at section C, but relative rotation and horizontal movement cannot occur. Therefore, the two sides of the incision remain parallel after the virtual displacements occur, and the relative vertical displacement of the incision is δ_Z . Based on the geometric relation of parallel lines, the values of critical points are determined as exhibited in figure 7.9d. Finally, the transformation of the virtual displacement diagram is performed. The parameter δ_Z in the ordinate is changed to 1, and the influence line for shear F_{SC} is attained as shown in figure 7.9e.

Example 7.3. Use the kinematic method to draw the influence lines for M_K , F_{SK} , M_C , F_{SE} , and F_{RD} of statically determinate multi-span beam shown in figure 7.10a.

Solution: (1) Construct the influence line for M_K .

The virtual displacements occur when a hinge is inserted to section K as shown in figure 7.10b. The relative rotational angle of two sides of hinge K is δ_Z , and the intercept is $BB' = 1 \cdot \delta_Z = \delta_Z$. The ordinates in figure 7.10b are enlarged and transformed, and the parameter δ_Z is changed to 1, so that the influence line for moment M_K is achieved as shown in figure 7.10c. The influence coefficients of critical points can be obtained in terms of the proportional relation. The figure of the influence line for M_K above the abscissa is positive, and the figure below the abscissa is negative.

(2) Construct the influence line for F_{SK} .

The virtual displacement diagram of the rigid body system (*i.e.*, the released system) is not drawn below, and the shape of the influence line is directly constructed.

The shape of the influence line for shear F_{SK} is proportional to the displacement δ_P diagram when the two sides of section K have vertical translation. First, remain the displacement of each support point to be zero when constructing the influence line. Then, the parallel lines AK' and $K''B$ are drawn on both sides of point K , and the translation at point K is the virtual displacement δ_Z corresponding to F_{SK} (its direction is consistent with the positive direction of shear F_{SK}). The parameter δ_Z in the ordinate is set as 1, that is, $K'K'' = 1$. Finally, the influence lines for auxiliary parts EF and FG are constructed. For this purpose, connect $E'C$ and extend to F' , then connect $F'D$ and extend to G' . Thus, the influence line for F_{SK} is obtained as shown in figure 7.10d.

(3) Construct the influence line for M_C .

After a hinge in section C is added, segments HE and EC still cannot produce virtual displacement. Thus, the influence line for moment M_C in segment HC overlaps with the baseline, but the auxiliary parts CF and FG can yield virtual displacement. The displacement δ_Z corresponding to M_C is the relative rotation of two segments of hinge C. Since EC has no rotation, the rotational angle of CF is δ_Z , and the vertical displacement of point F is $2 \cdot \delta_Z$. Change the parameter δ_Z in the

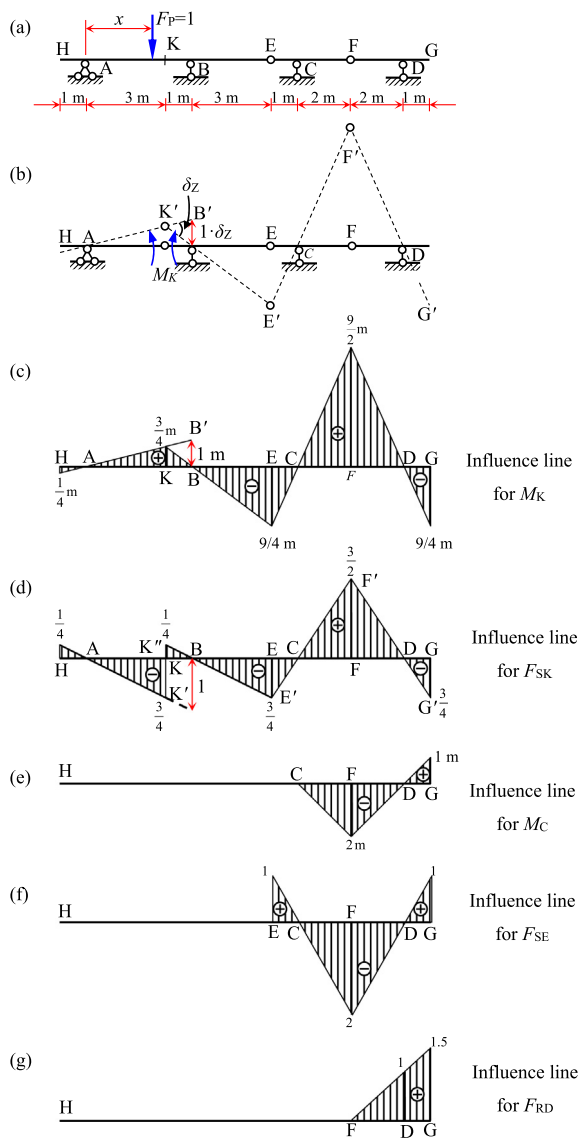


FIG. 7.10 – Influence lines for internal forces and reaction of statically determinate multi-span beam by kinematic method.

ordinate to 1, and thus the influence coefficient of point F is equal to 2 m. The influence line for M_C is shown in figure 7.10e.

(4) Construct the influence line for F_{SE} .

When the two sides of section E have vertical translation in the positive direction of F_{SE} , the basic part of HE cannot yield displacement. Therefore, the influence line

for F_{SE} in segment HE is always equal to zero. The segment EF rotates around support C, and the segment FG rotates around support D. The relative displacement δ_Z (ordinate of point E) corresponding to shear F_{SE} is changed to 1, and the influence line for F_{SE} is constructed as shown in figure 7.10f.

(5) Construct the influence line for F_{RD} .

In the statically determinate multi-span beam, FG is an auxiliary part of HF. When support D is removed, segment HF still cannot generate displacement, so the influence line for reaction F_{RD} in segment HF always equals zero. The vertical displacement of segment FG along the direction of F_{RD} at point D is δ_Z , then let $\delta_Z = 1$, and the influence line for F_{RD} is drawn in figure 7.10g.

It is seen from the diagrams of various influence lines shown in figure 7.10 that in the statically determinate multi-span beam, the influence lines for internal force (or support reaction) of the basic part are full of the entire beam. Nevertheless, for the influence lines of auxiliary parts, only the influence coefficients of auxiliary parts are not zero (the influence coefficients of the basic parts are constantly zero), and the influence ranges of moving load on the internal force or support reaction of the basic parts and auxiliary parts are different. This conclusion is consistent with the mechanical properties of statically determinate multi-span beams.

Example 7.4. Apply the kinematic method to construct the influence lines for M_K , F_{SK} , F_{SC}^L and F_{yF} of the girder system under joint loads shown in figure 7.11a. The spacing of floor beams is 2 m.

Solution: In this example, the girder is subjected to joint load. It is noted that the displacement diagram $\delta_P(x)$ in the kinematic method refers to the displacement diagram of the applying point of unit moving load. Therefore, when the kinematic method is used to construct the influence line for statically determinate force under joint load, $\delta_P(x)$ should be the displacement diagram of the stringers rather than girders. It is because the unit load is moving on the stringers. The diagram range of influence lines of the internal force or support reaction for the girders should be the range that the moving load can reach, namely, from point A to point I. When using the kinematic method, the corresponding influence lines (point A to point H) under direct load can be drawn separately, and the region of broken lines between the floor beams (*i.e.*, joints) is connected by straight lines, and then the diagram is extended to point I. Finally, the ordinates of critical points are determined according to the geometric relation. The various influence lines obtained are shown in figure 7.11b–e.

7.5 Applications of Influence Lines

The influence line is the basic tool for structural analysis under moving loads. The influence line can be used to determine the changing regulation of structural response (reaction, shear, or moment) with the acting position of loads, and the magnitudes of internal forces of structures due to various kinds of loads, as well as

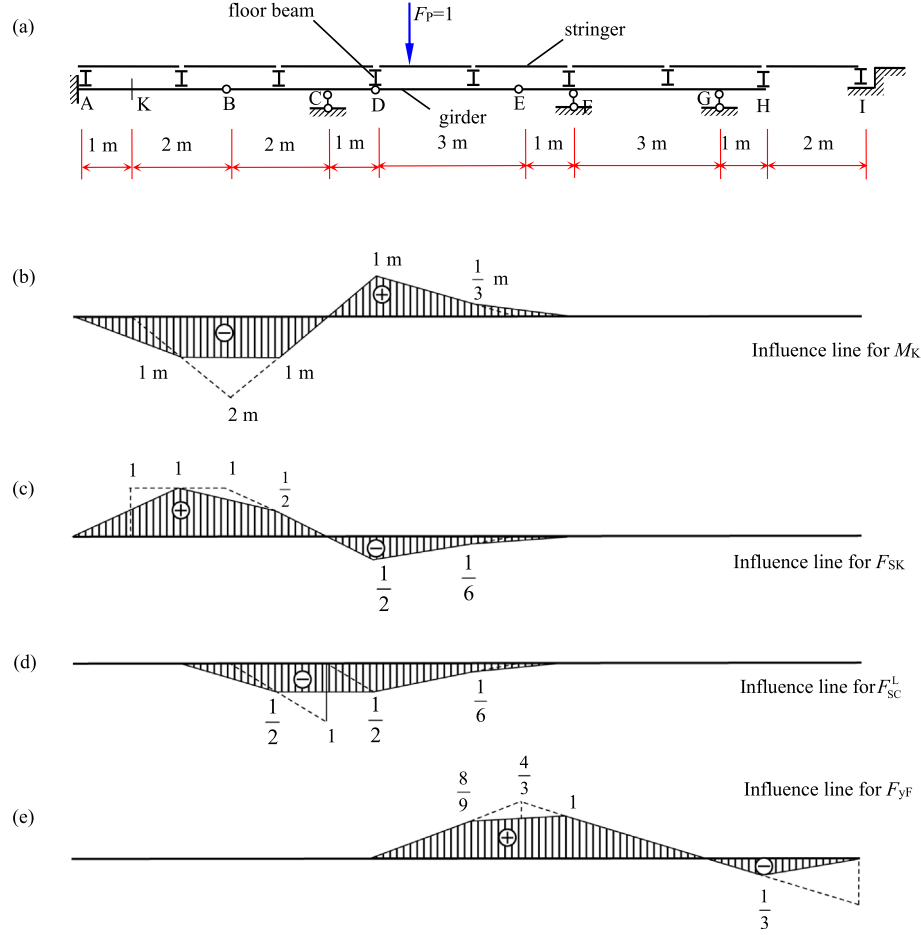


FIG. 7.11 – Influence lines for internal forces of girder system under joint loads by kinematic method.

the most unfavorable positions of structures under uniformly distributed live load or a series of moving concentrated loads.

7.5.1 Responses Due to Various Kinds of Loads

Actually, the unit moving load is employed to determine the influence lines for internal forces and support reactions of statically determinate and indeterminate structures. According to the superposition principle, the influence line can be utilized to compute the magnitudes of structural responses subjected to various kinds of loads.

A set of concentrated loads F_{P1} , F_{P2} , and F_{P3} are applied on the simply supported beam, and the positions are known as shown in figure 7.12a. If the ordinates

of the influence line (figure 7.12c) for shear F_{SC} at loading points are y_1, y_2, y_3 , then F_{SC} caused by F_{P1} is equal to $F_{P1} y_1$, F_{SC} caused by F_{P2} is equal to $F_{P2} y_2$, and F_{SC} caused by F_{P3} is equal to $F_{P3} y_3$. In terms of the superposition principle, the magnitude of F_{SC} under this set of loads is

$$F_{SC} = F_{P1} y_1 + F_{P2} y_2 + F_{P3} y_3$$

Generally speaking, a set of concentrated loads $F_{P1}, F_{P2}, \dots, F_{Pn}$ are applied on the structure, and the ordinates of influence line for a structural response Z at various loading points are $y_1, y_2, \dots, y_n$

$$Z = F_{P1} y_1 + F_{P2} y_2 + \dots + F_{Pn} y_n = \sum_{i=1}^n F_{Pi} y_i \quad (7.4)$$

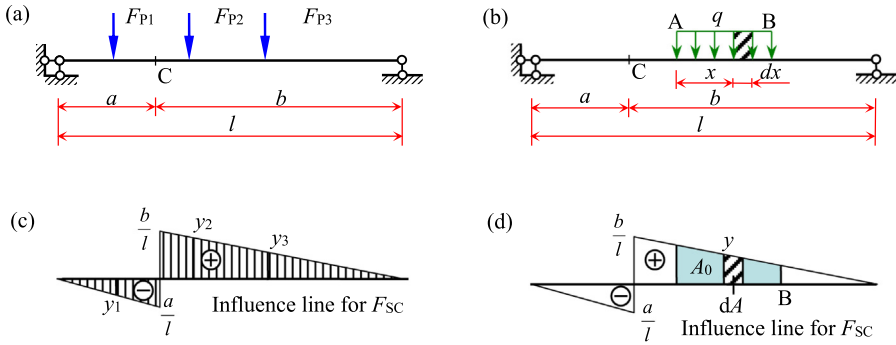


FIG. 7.12 – Simply supported beam under concentrated and distributed loads and influence line of shear F_{SC} .

If the structure has uniformly distributed load q over the segment AB (figure 7.12b), the load qdx at the differential element of length dx can be regarded as a concentrated load, and the magnitude of Z caused by it is $y \cdot qdx$. Thus, the magnitude of Z under uniformly distributed load cross segment AB is

$$Z = \int_A^B y q dx = q \int_A^B y dx = q A_0 \quad (7.5)$$

where the area of influence line diagram (figure 7.12d) over the loading segment AB is indicated by A_0 . Equation (7.5) denotes that the magnitude of Z caused by the uniformly distributed load is equal to the load intensity multiplied by the influence line area of the loading segment. When applying this formula, we should notice the sign of area A_0 .

Example 7.5. Figure 7.13 illustrates a simply supported beam with a full span subjected to the uniformly distributed load. Calculate the magnitude of shear F_{SC} by using the influence line for F_{SC} of section C.

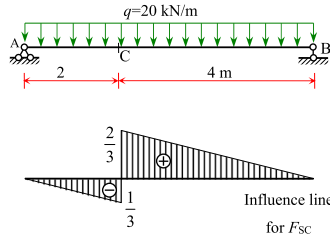


FIG. 7.13 – Calculation of shear F_{SC} of beam using influence line.

Solution: The area of plus sign part of the influence line of F_{SC} is expressed by A_1 , and the area of minus sign part is denoted by A_2 , thus

$$A_1 = \frac{1}{2} \times \frac{2}{3} \times 4 \text{ m} = \frac{4}{3} \text{ m}$$

$$A_2 = \frac{1}{2} \times \left(-\frac{1}{3}\right) \times 2 \text{ m} = -\frac{1}{3} \text{ m}$$

According to equation (7.5), we have

$$F_{SC} = q(A_1 + A_2) = 20 \text{ kN/m} \times \left(\frac{4}{3} \text{ m} - \frac{1}{3} \text{ m}\right) = 20 \text{ kN}$$

7.5.2 Most Unfavorable Position of Moving Loads

If the loads move to a certain position, the response Z of the structure reaches the maximum, and such a loading position is called the most unfavorable loading position for Z . An important role of the influence line is to determine **the most unfavorable position of loads**.

For some simple cases, the most unfavorable position of loads can be determined by some analysis and judgment on the influence line and loading characteristics. The general principle of judgment is that the loads with large magnitude and close arrangement should be placed on the location with great ordinate of influence line. Here, a few simple examples are taken as follows.

If the moving load is a single concentrated load, the corresponding most unfavorable position is the position where this concentrated load is acting at the maximum ordinate of the influence line.

If the moving loads are a set of concentrated loads, then at the most unfavorable position, there must be a concentrated load acting at the vertex of the influence line (figure 7.14). The next subsection will elaborate on the reason.

If the moving loads are uniformly distributed and can be arranged in an arbitrary way, the corresponding most unfavorable position is that the plus sign part of the influence line is full of loads (to solve the maximum positive value of response), or the minus sign part is full of loads (to solve the maximum negative value of response) (figure 7.15).

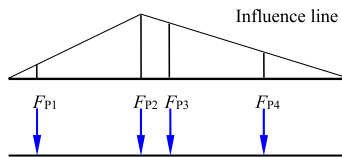


FIG. 7.14 – Most unfavorable position of a set of concentrated loads.

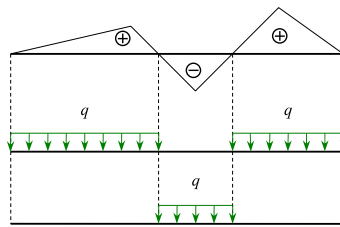


FIG. 7.15 – Most unfavorable position of uniformly distributed loads.

Example 7.6. Figure 7.16a demonstrates the wheel pressure and wheel span of two cranes, and calculate the maximal shear of section C in the crane beam AB.

Solution: First, draw the influence line for shear F_{SC} in the crane beam (figure 7.16c).

To maximize the shear F_{SC} , the live loads should first be positioned on the plus sign portion of the influence line for F_{SC} . Secondly, the loads with the close arrangement (the middle two wheel pressures) should be placed at the location with a larger influence coefficient (*i.e.*, the load of 435 kN is placed at the right side of section C). Figure 7.16b displays the most unfavorable position of loads. Thus, the maximum shear of section C is

$$F_{SC_{\max}} = F_{P1}y_1 + F_{P2}y_2 = 435 \text{ kN} \times 0.667 + 295 \text{ kN} \times 0.425 = 415 \text{ kN}$$

Example 7.7. The simply supported beam in figure 7.13 is subjected to a uniformly distributed load with $q = 20 \text{ kN/m}$, and the load can be arbitrarily arranged on the beam. Solve the maximal and minimal values of shear F_{SC} .

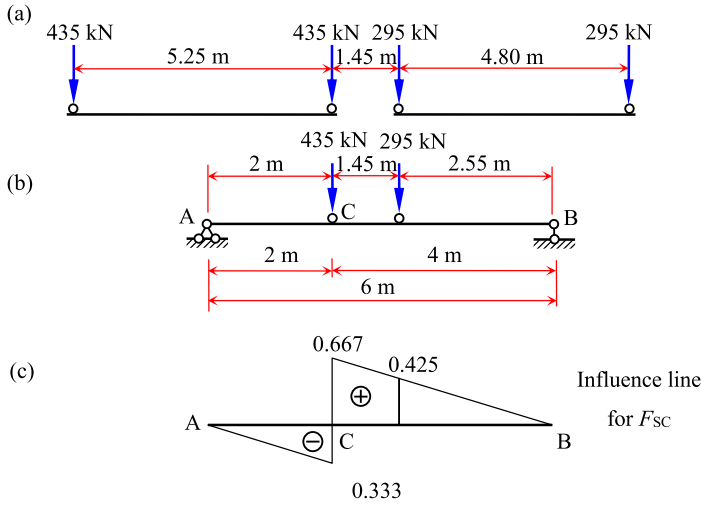


FIG. 7.16 – (a) Wheel pressure and wheel span of two cranes, (b) the most unfavorable position of loads, (c) influence line for shear F_{SC} of crane beam AB.

Solution: It is seen from the influence line for F_{SC} shown in figure 7.13 that, the maximal shear can be obtained when the loads are placed in the whole segment CB

$$F_{SC\max} = qA_{CB} = 20 \text{ kN/m} \times \frac{4}{3} \text{ m} = 26.7 \text{ kN}$$

The minimal shear is obtained when the loads are placed in the whole segment AC

$$F_{SC\min} = qA_{AC} = 20 \text{ kN/m} \times \left(-\frac{1}{3} \text{ m}\right) = -6.7 \text{ kN}$$

7.5.3 Determination of Critical Position for Polygonal Influence Line

If the moving load is a series of concentrated loads, the most unfavorable position of loads for a certain structural response Z can be usually determined in terms of the following two steps.

In the first step, determine the loading position where Z reaches the extreme value. Such a loading position is named the **critical position** of loads.

In the second step, the most unfavorable position of loads is selected from the critical position of loads, that is, the maximum is selected from the extremely large values of response Z , and the minimum is chosen from the extremely small values.

The polygonal influence line of structural response is taken as an example to illustrate the feature and determination method of the critical loading position.

Figure 7.17a indicates a set of concentrated loads. Their spacing and magnitudes remain invariable as the loads move. The influence line for response Z shown in figure 7.17b is a polygon. The slopes of the sides are represented by α_1 , α_2 , and α_3 separately (where α_1 and α_2 are positive and α_3 is negative). The resultant forces of concentrated loads at the interval of each side are denoted by F_{R1} , F_{R2} , and F_{R3} , respectively.

According to the superposition principle and the resultant forces of loads at the interval of each side, we have

$$Z = F_{R1}\bar{y}_1 + F_{R2}\bar{y}_2 + F_{R3}\bar{y}_3 = \sum_{i=1}^3 F_{Ri}\bar{y}_i$$

where $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ are the influence coefficients corresponding to the resultant forces F_{R1} , F_{R2} and F_{R3} of loads of various sides.

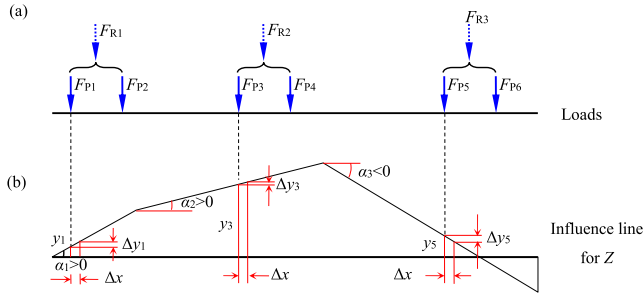


FIG. 7.17 – (a) A set of concentrated loads, (b) polygonal influence line for response Z .

Assuming that the loads move a distance Δx (Δx is positive when moving to the right), the increment of influence coefficient $\bar{y}_i$ is

$$\Delta\bar{y}_i = \Delta x \cdot \tan \alpha_i$$

and the increment of Z is

$$\Delta Z = \Delta x \cdot \sum_{i=1}^3 F_{Ri} \tan \alpha_i$$

For the critical loading position where the response Z reaches the extreme large value, it is necessary to satisfy the following condition: when the loads move to the right or left from the critical position, the magnitude of Z should decrease, $\Delta Z < 0$, namely

$$\Delta x \cdot \sum_{i=1}^3 F_{Ri} \tan \alpha_i < 0 \quad (a)$$

Equation (a) can also be divided into two cases:

$$\begin{cases} \text{if } \Delta x > 0 \text{ (loads move slightly to the right), } & \sum F_{Ri} \tan \alpha_i < 0 \\ \text{if } \Delta x < 0 \text{ (loads move slightly to the left), } & \sum F_{Ri} \tan \alpha_i > 0 \end{cases} \quad (7.6)$$

Similarly, for the critical loading position where the response Z reaches the extreme small value, the following condition must be met:

$$\begin{cases} \text{if } \Delta x > 0 \text{ (loads move slightly to the right), } & \sum F_{Ri} \tan \alpha_i > 0 \\ \text{if } \Delta x < 0 \text{ (loads move slightly to the left), } & \sum F_{Ri} \tan \alpha_i < 0 \end{cases} \quad (7.7)$$

At this time, the following conclusion can be drawn: if Z is the extreme value (extremely large or extremely small), the sign of $\sum F_{Ri} \tan \alpha_i$ must be changed when the loads move slightly to the left or right. The term $\sum F_{Ri} \tan \alpha_i$ is called as the determination formula of critical position.

Under what situations might the determination formula $\sum F_{Ri} \tan \alpha_i$ change the sign? $\tan \alpha_i$ in the formula is the slope of a straight line of each segment in the influencing line, which is a constant. Therefore, only the resultant force F_{Ri} within each segment changes its magnitude, it is possible to change the sign of $\sum F_{Ri} \tan \alpha_i$ when the entire loads move slightly to the left or right. Obviously, in the critical position, there must be a concentrated load acting exactly at the vertex of the influence line. For example, if the concentrated force F_{Ri} is applied at the vertex between the straight line of the i th segment and that of the $i + 1$ th segment, then F_{Pcr} should be counted into F_{Ri} when the entire loads slightly move to the left, and F_{Pcr} should be counted into F_{Ri} when the entire loads slightly move to the right. In a word, when the loads move slightly to the left or right, the necessary condition to change the sign of $\sum F_{Ri} \tan \alpha_i$ is a concentrated load acting at the vertex of the influence line, but this is not a sufficient condition.

To sum up, the procedure for determining the most unfavorable position of moving loads is described as follows.

- (1) Select a concentrated force F_{Pcr} from the moving loads and position it at one of the vertexes of the influence line.
- (2) When F_{Pcr} moves slightly to the left or right at the vertex, the values of $\sum F_{Ri} \tan \alpha_i$ are obtained separately. If the sign of $\sum F_{Ri} \tan \alpha_i$ is changed (or changed from zero to non-zero), then this loading position is called critical position, and the load F_{Pcr} is called **critical load**. If the sign of $\sum F_{Ri} \tan \alpha_i$ is invariable, this loading position is not a critical position.
- (3) An extreme value of Z can be obtained for each critical position, and then the maximum or minimum is selected from various extreme values. Meantime, the most unfavorable position of loads is also determined.

Example 7.8. Figure 7.18a shows a set of moving loads for railway train. Five concentrated forces correspond to the 5 axle weights of one locomotive, and the average intensity of the uniformly distributed load of length 30 m for another locomotive in the middle of loads is q_1 , and the average intensity of a train vehicle of arbitrary length in the rear is q_2 . The influence line for a certain structural response

Z is shown in figure 7.18b. When the train travels from the right to left, determine the most unfavorable position of loads and the maximum value of Z . It is known that $F_{P1} = F_{P2} = F_{P3} = F_{P4} = F_{P5} = 220$ kN, $q_1 = 92$ kN/m, and $q_2 = 80$ kN/m.

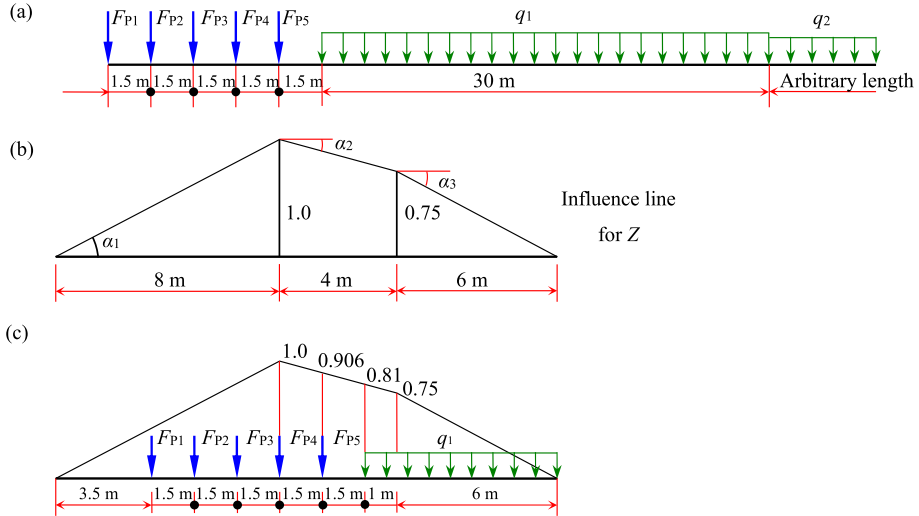


FIG. 7.18 – (a) A set of moving loads for railway train, (b) influence line of response Z , (c) loading layout.

Solution: (1) The loads of the locomotive are dense and large. It can be judged that the most unfavorable position of loads is to place the load F_{P4} at the highest vertex of influence line. The loading layout is exhibited in figure 7.18c.

(2) The determination formula (7.6) for critical position of loads is adopted for check.

In terms of figure 7.18b, we have

$$\tan \alpha_1 = \frac{1}{8}, \tan \alpha_2 = -\frac{0.25}{4}, \tan \alpha_3 = -\frac{0.75}{6}$$

If the entire loads move slightly to the right, the resultant forces of three segments corresponding to the influence line are

$$F_{R1} = 220 \text{ kN} \times 3 = 660 \text{ kN}$$

$$F_{R2} = 220 \text{ kN} \times 2 + 92 \text{ kN/m} \times 1 \text{ m} = 532 \text{ kN}$$

$$F_{R3} = 92 \text{ kN/m} \times 6 \text{ m} = 552 \text{ kN}$$

Therefore

$$\begin{aligned}\sum F_{Ri} \tan \alpha_i &= 660 \text{ kN} \times \frac{1}{8} + 532 \text{ kN} \times \left(-\frac{0.25}{4}\right) + 552 \text{ kN} \times \left(-\frac{0.75}{6}\right) \\ &= -19.75 \text{ kN} < 0\end{aligned}$$

If the entire loads move slightly to the left, we obtain

$$F_{R1} = 220 \text{ kN} \times 4 = 880 \text{ kN}$$

$$F_{R2} = 220 \text{ kN} + 92 \text{ kN/m} \times 1 \text{ m} = 312 \text{ kN}$$

$$F_{R3} = 92 \text{ kN/m} \times 6 \text{ m} = 552 \text{ kN}$$

$$\begin{aligned}\sum F_{Ri} \tan \alpha_i &= 880 \text{ kN} \times \frac{1}{8} + 312 \text{ kN} \times \left(-\frac{0.25}{4}\right) + 552 \text{ kN} \times \left(-\frac{0.75}{6}\right) \\ &= 21.5 \text{ kN} > 0\end{aligned}$$

This position is a critical position because the sign of $\sum F_{Ri} \tan \alpha_i$ changes.

(3) Calculate the maximum of Z .

Using the influence coefficients marked in figure 7.18c, we have

$$\begin{aligned}Z_{\max} &= 220 \text{ kN} \times \left(\frac{3.5}{8} + \frac{5}{8} + \frac{6.5}{8} + 1\right) + 220 \text{ kN} \times 0.906 + 92 \text{ kN/m} \\ &\quad \times \left(\frac{0.81 + 0.75}{2} \times 1 \text{ m} + \frac{0.75 \times 6 \text{ m}}{2}\right) = 1110.58 \text{ kN}\end{aligned}$$

7.5.4 Determination of Critical Position for Triangle Influence Line

When the influence line is a triangle, the characteristic of the critical position of loads can be expressed in a more convenient form. As shown in figure 7.19, assume the influence line for response Z is a triangle. If the maximal value of Z is to be determined, there must be a load F_{Pcr} just acting at the vertex of the influence line. F_{R}^{L} represents the resultant force for the left loads of F_{Pcr} , and F_{R}^{R} stands for the resultant force for the right loads of F_{Pcr} . To maximize the response Z , the determination formula (7.6) for the critical loading position can be written as

$$\text{Loads move to the right : } F_{\text{R}}^{\text{L}} \tan \alpha - (F_{\text{Pcr}} + F_{\text{R}}^{\text{R}}) \tan \beta < 0$$

$$\text{Loads move to the left : } (F_{\text{R}}^{\text{L}} + F_{\text{Pcr}}) \tan \alpha - F_{\text{R}}^{\text{R}} \tan \beta > 0$$

substituting $\tan \alpha = \frac{c}{a}$ and $\tan \beta = \frac{c}{b}$ into the above formula yields

$$\begin{cases} \frac{F_R^L}{a} < \frac{F_{Pcr} + F_R^R}{b} \\ \frac{F_R^L + F_{Pcr}}{a} > \frac{F_R^R}{b} \end{cases} \quad (7.8)$$

The two sides of inequality (7.8) can be understood as the “average load” of each side. It is seen that for the triangle influence line, the characteristics of the critical position and critical load F_{Pcr} can be summarized as follows.

- (1) There is a concentrated load F_{Pcr} exactly acting at the vertex of triangle influence line.
- (2) This concentrated load F_{Pcr} plays an important role. If it moves towards the left of the vertex, the left side of the influence line has more load; while if it moves towards the right, the right side has more load (for average load).

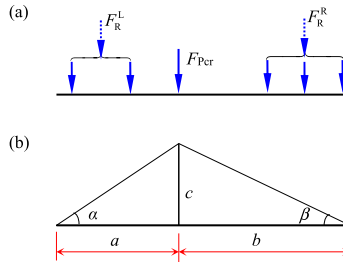


FIG. 7.19 – Critical position of loads for triangle influence line.

Example 7.9. Figure 7.20a illustrates a set of motorcade loads, one of which is a heavy motor and the rest are regular motors. Figure 7.20b shows a simply supported beam bridge AB with a span of length 40 m. Calculate the maximum bending moment of section C in this beam under loads of the vehicle.

Solution: At the beginning, we explain two points.

- (1) This motorcade has two ways to pass this beam bridge, namely, pass from right to left (figure 7.20a) or from left to right (figure 7.21a).
- (2) According to the intuitive judgment, the heavy motors with the largest resultant force must be placed near the vertex of the influence line, and certain concentrated loads should be applied at the vertex, so that the most unfavorable position of live loads can be determined basically. For further check, the determination formula (7.8) critical position can be adopted.

The influence line for moment M_C is a triangle as shown in figure 7.20c.

Firstly, suppose the motorcade passes the bridge from the right to left. The rear wheel of heavy motor of motorcade shown in figure 7.20a, that is, the fifth axle

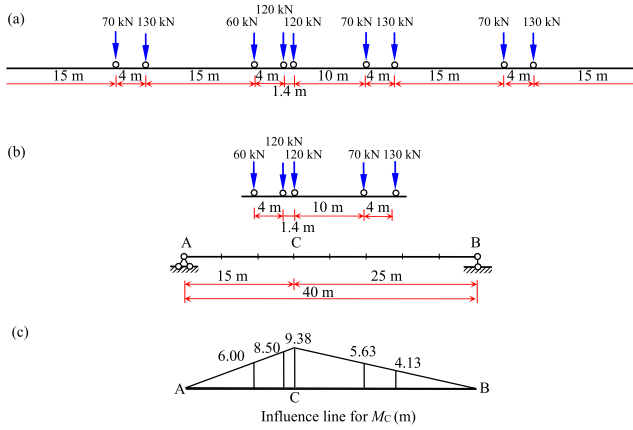


FIG. 7.20 – (a) Motorcade passing beam bridge from right to left, (b) the most unfavorable position of live loads, (c) influence line for moment M_C .

weight 120 kN from the left of the loads is placed at point C (figure 7.20b). The determination formula (7.8) is applied to check as follows:

$$\frac{60 \text{ kN} + 120 \text{ kN}}{15 \text{ m}} < \frac{60 \text{ kN} + 70 \text{ kN} + 130 \text{ kN}}{25 \text{ m}}$$

$$\frac{60 \text{ kN} + 120 \text{ kN} + 120 \text{ kN}}{15 \text{ m}} > \frac{70 \text{ kN} + 130 \text{ kN}}{25 \text{ m}}$$

The position is a critical position and the corresponding value of M_C is

$$M_C = 60 \text{ kN} \times 6.00 \text{ m} + 120 \text{ kN} \times 8.50 \text{ m} + 120 \text{ kN} \times 9.38 \text{ m} + 70 \text{ kN} \times 5.63 \text{ m} + 130 \text{ kN} \times 4.13 \text{ m} = 3436.6 \text{ kN}\cdot\text{m}$$

Secondly, assume the motorcade passes the bridge from the left to right (figure 7.21a). The rear wheel of heavy motor of the motorcade, that is, the fifth axis weight 120 kN from the right of the loads is applied at point C (figure 7.21b), and the determination formula (7.8) is used to check:

$$\frac{130 \text{ kN} + 70 \text{ kN}}{15 \text{ m}} < \frac{120 \text{ kN} + 120 \text{ kN} + 60 \text{ kN} + 130 \text{ kN} + 70 \text{ kN}}{25 \text{ m}}$$

$$\frac{130 \text{ kN} + 70 \text{ kN} + 120 \text{ kN}}{15 \text{ m}} > \frac{120 \text{ kN} + 60 \text{ kN} + 130 \text{ kN} + 70 \text{ kN}}{25 \text{ m}}$$

This position is also a critical position and the corresponding moment M_C is

$$M_C = 130 \text{ kN} \times 0.63 \text{ m} + 70 \text{ kN} \times 3.13 \text{ m} + 120 \text{ kN} \times 9.38 \text{ m} + 120 \text{ kN} \times 8.85 \text{ m} + 60 \text{ kN} \times 7.35 \text{ m} + 130 \text{ kN} \times 1.73 \text{ m} + 70 \text{ kN} \times 0.23 \text{ m} = 3170.6 \text{ kN}\cdot\text{m}$$

Comparing the above calculated results, it is known that the loading position shown in figure 7.20b is the most unfavorable position. The maximal moment of section C is 3436.6 kN·m.

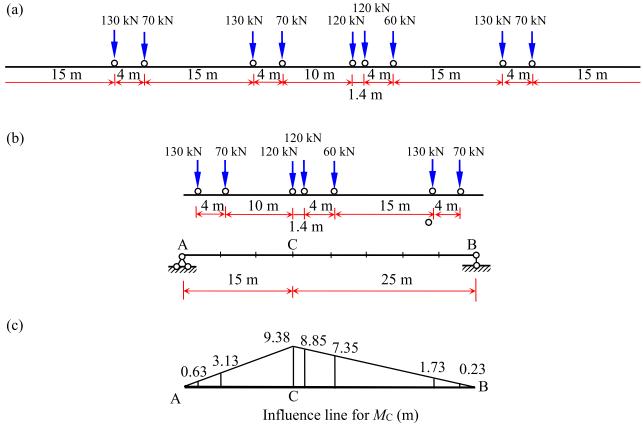


FIG. 7.21 – (a) Motorcade passing beam bridge from left to right, (b) the most unfavorable position of loads, (c) influence line for moment M_C .

7.6 Kinematic Method for Constructing Influence Lines of Statically Indeterminate Beams

First of all, according to the principle of virtual displacements for a deformable body, the basic formulation for constructing the influence line of internal force or reaction of the statically indeterminate structure is derived. This formulation is the same as the basic formulation for force influence line of the statically determinate structure by the kinematic method. Moreover, a simple and convenient method is established, which can exactly and accurately calculate the influence line equations of support reactions and internal forces of statically indeterminate structures.

The influence line for response Z_1 standing for the reaction and internal force of statically indeterminate structure is considered to construct. On the one hand, the force state of the original structure under unit moving load $F_P = 1$ displayed in figure 7.22a is equivalent to that of the released structure by relaxing relevant constraint with constraint force (generalized load) Z_1 in figure 7.22b. Its support reaction is F_R , and the internal forces are $M(x)$, $F_N(x)$, and $F_S(x)$. The generalized load Z_1 of the released structure in figure 7.22b is equal to the corresponding internal force of the original structure, and the dashed lines in figure 7.22 are the deflected curves, which are identical to the deflected curves of the original structure. Here, the force states and the displacement states of the original structure and the released structure are the same. On the other hand, assume that the released structure is exerted by a unit generalized displacement $\bar{c}_1 = 1$ relevant to the relaxed

constraint, and there are no other support displacements. Then, the released structure yields the virtual displacement $\bar{\delta}_P$ at the point corresponding to the unit moving load and the virtual strains $\bar{\kappa}$, $\bar{\varepsilon}$ and $\bar{\gamma}_0$ shown in figure 7.22c.

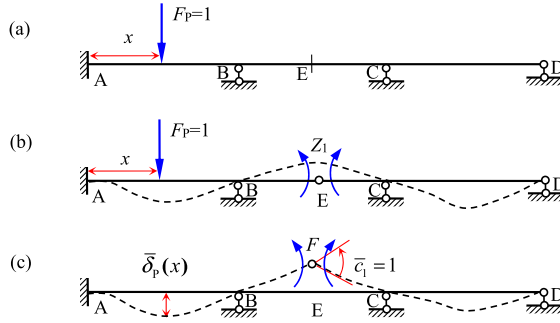


FIG. 7.22 – (a) Original structure, (b) released structure, (c) released structure with unit generalized displacement.

Substituting above two states into the virtual displacement equation of deformable body yields:

$$Z_1 \cdot 1 + 1 \cdot \bar{\delta}_P = \sum \int (M\bar{\kappa} + F_N\bar{\varepsilon} + F_S\bar{\gamma}_0)dx \quad (7.9)$$

It is known from the analysis that the right-hand term of equation (7.9) is zero, that is, the internal virtual work done by the internal forces of statically indeterminate structure under unit moving load and generalized load undergoing the virtual strains of its released structure caused by virtual unit displacement in the direction of respective restraint is zero. Therefore, equation (7.9) becomes

$$Z_1(x) = -\bar{\delta}_P(x) \quad (7.10)$$

Equation (7.10) is the basic formulation for drawing influence lines for reactions and internal forces of the indeterminate structure *via* the kinematic method. It means that the influence coefficients $Z_1(x)$ of internal forces or reactions of the original structure are exactly equal to the virtual deflections of the released structure with unit generalized displacement at the acting point of unit moving load, but its sign is opposite. Clearly, the formulation of the kinematic method for constructing the influence lines for reactions and internal forces of statically indeterminate structure derived from the principle of virtual displacements is consistent with that of statically determinate structure.

It is worth noting that the basic formulation equation (7.10) of the kinematic method can also be derived from the theorem of reciprocal works. Under the unit moving load $F_P = 1$, the indeterminate structure generates some deformations and internal forces, for example, as displayed in figure 7.22a. The equivalent released structure of this state is defined as state I (figure 7.22b). This state is equivalent to

relaxing the corresponding constraint at Z_1 , and acting the external force Z_1 equal to the constraint force of the original structure. Meanwhile, the state of forces and deformations of the released structure caused by a generalized displacement in the direction of relaxed constraint in figure 7.22c is defined as state II.

The external forces of state I include the unit moving load $F_P = 1$ and generalized load Z_1 . The displacements of state II include the unit generalized displacement $\bar{c}_1 = 1$ (without support displacement) and the displacement $\bar{\delta}_P$ along the direction of the unit moving load. According to the reciprocal theorem of works, we have

$$W_{12} = W_{21}$$

The work done by external forces of state I undergoing the displacements of state II is: $W_{12} = 1 \cdot \bar{\delta}_P + Z_1 \cdot 1$. If the external forces of state II are 0 (when the released structure is statically indeterminate structure, and the support displacement is caused by the support settlement, then the external force is 0), and the support displacements of state I corresponding to support reactions of state II are 0, then $W_{21} = 0$. If the external forces of state II are not 0 (for example, the corresponding generalized displacement of the statically indeterminate structure shown in figure 7.22c is caused by the external force F), but the relative displacement corresponding to a pair of non-zero external forces in the original structure is 0 (the generalized displacement of the left and right sides of section E in figure 7.22b, *i.e.*, the relative rotation is zero), then there is still $W_{21} = F \times 0 = 0$. Thus

$$W_{12} = 1 \cdot \bar{\delta}_P + Z_1 \cdot 1 = W_{21} = 0 \quad (7.11)$$

In terms of equation (7.11), the influence line formula equation (7.10) of structural reaction and internal force can be obtained.

According to the formulation of equation (7.10), the process of the kinematic method to construct the influence lines for reactions and internal forces of statically indeterminate structures is proposed as follows.

- (1) In order to determine the influence line for internal force or reaction force Z of the statically indeterminate structure, we remove the constraint corresponding to the internal force (reaction force) Z to form the released structure. Impose the corresponding generalized unit displacements $\delta_Z = 1$ corresponding to the relaxed constraint of released structure, and take the deflected curve as the shape of the influence line for the statically indeterminate force.
- (2) Utilize the load-displacement differential relationship of the prismatic beam with bending stiffness EI :

$$EIy^{(4)}(x) = q(x)$$

Since the released structure has no distributed loads, namely $q(x) = 0$, the analytical expression of the deflection curve of each segment for the released structure is a cubic polynomial function

$$y(x) = ax^3 + bx^2 + cx + d$$

Considering the specific cases of the displacement and force of segments in beam-like structures, four boundary and continuity conditions for every segment can be written.

- (3) Substituting the four boundary and continuity conditions for every segment of the released structure into the deflection curve function with 4 undetermined coefficients and its first, second, and third-order differential formulas, connect the solved deflection curves of segments to achieve the displacement curve $y(x)$ of the released structure. Consequently, the analytical influence line equation $Z(x) = y(x)$ of the statically indeterminate internal force (reaction force) can be quickly obtained.

Finally, construct the curve diagram of the cubic function according to the analytical equation of the influence line. The influence coefficient above the abscissa is taken as a plus sign, and that below the abscissa is taken as a minus sign. On this basis, the most unfavorable position of loads and the most unfavorable internal forces of indeterminate structure can be determined.

Example 7.10. Draw the influence lines for shear F_{SD} and moment M_D at section D of the two-span statically indeterminate continuous beam shown in figure 7.23.

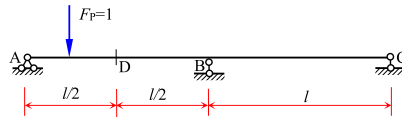


FIG. 7.23 – Two-span statically indeterminate continuous beam.

Solution: (1) Construct the influence line for F_{SD} .

Relax the section D of original structure with a sliding device, and obtain the released structure with three segments. Subsequently, impose the unit relative translation $\delta_D = 1$ at D, and outline the influence line shape of shear F_{SD} according to the virtual displacement diagram in figure 7.24.

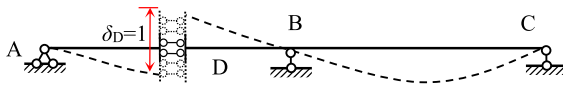


FIG. 7.24 – Shape of influence line for shear F_{SD} .

Three cubic polynomials $y_i(x) = ax^3 + bx^2 + cx + d$ of released structure are written, which contain 12 undetermined coefficients. Twelve boundary and continuity conditions are listed as:

$$y_1(0) = y_2(l) = y_3(l) = y_3(2l) = 0, \quad y_2(l/2) - y_1(l/2) = 1, \quad y'_1(l/2) = y'_2(l/2),$$

$$y'_2(l) = y'_3(l), \quad EIy''_1(0) = EIy''_3(2l) = 0, \quad EIy''_2(l) = EIy''_3(l),$$

$$EIy''_1(l/2) = EIy''_2(l/2), \quad EIy'''_1(l/2) = EIy'''_2(l/2)$$

After calculating the undetermined coefficients by solving simultaneous linear algebraic equations, the influence line equation for F_{SD} can be obtained

$$y(x) = \begin{cases} \frac{1}{4l^3}x^3 - \frac{5}{4l}x & (0 \leq x \leq l/2) \\ \frac{1}{4l^3}x^3 - \frac{5}{4l}x + 1 & (l/2 \leq x \leq l) \\ -\frac{1}{4l^3}x^3 + \frac{3}{2l^2}x^2 - \frac{11}{4l}x + \frac{3}{2} & (l \leq x \leq 2l) \end{cases}$$

(2) Construct the influence line for M_D .

Relax the section D with a hinge to get the released structure with three segments, and then impose the unit relative rotation $\theta_D = 1$ at hinge D and outline the influence line shape of moment M_D according to the deflected curve, as shown in figure 7.25.

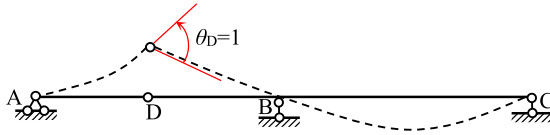


FIG. 7.25 – Shape of influence line for moment M_D .

Similarly, three cubic polynomials $y_i(x) = ax^3 + bx^2 + cx + d$ of the released structure with 12 unknown coefficients are listed, and 12 boundary and continuity conditions are written as:

$$y_1(0) = y_2(l) = y_3(l) = y_3(2l) = 0, \quad y_1(l/2) = y_2(l/2),$$

$$y'_1(l/2) - y'_2(l/2) = 1, \quad y'_2(l) = y'_3(l), \quad EIy''_2(l) = EIy''_3(l),$$

$$EIy''_1(0) = EIy''_3(2l) = 0, \quad EIy''_1(l/2) = EIy''_2(l/2), \quad EIy'''_1(l/2) = EIy'''_2(l/2)$$

The unknown coefficients are solved, and the influence line equation for M_D is formulated

$$y(x) = \begin{cases} \frac{1}{8l^2}x^3 + \frac{3}{8}x & (0 \leq x \leq \frac{l}{2}) \\ \frac{1}{8l^2}x^3 - \frac{5}{8}x + \frac{l}{2} & (\frac{l}{2} \leq x \leq l) \\ -\frac{1}{8l^2}x^3 + \frac{3}{4l}x^2 - \frac{11}{8}x + \frac{3l}{4} & (l \leq x \leq 2l) \end{cases}$$

Finally, take $l = 6$ m and draw the influence lines for F_{SD} and M_D shown in figure 7.26.

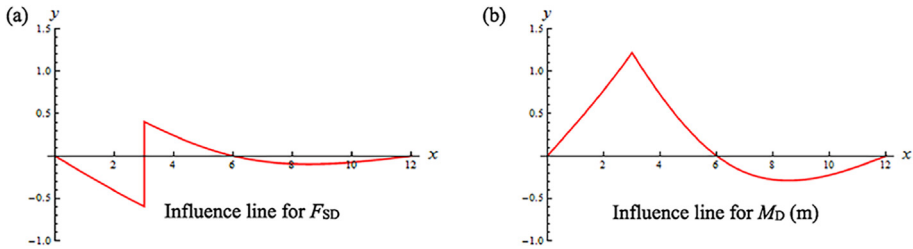


FIG. 7.26 – Influence line for F_{SD} and M_D of two-span statically indeterminate continuous beam.

Example 7.11. Construct the influence lines for support reaction F_{RB} , moment M_B and moment M_E at section E of the statically indeterminate beam with 3 degrees of indeterminacy in figure 7.27.

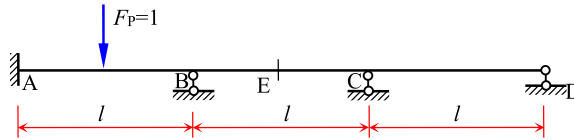


FIG. 7.27 – Statically indeterminate beam with 3 degrees of indeterminacy.

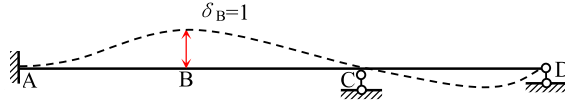
Solution: (1) Construct the influence line for F_{RB} .

Remove the support B and obtain the released structure with three segments. Then, impose unit displacement $\delta_B = 1$ at B, and outline the influence line shape of reaction F_{RB} in figure 7.28.

Three cubic polynomials $y_i(x) = ax^3 + bx^2 + cx + d$ of the released structure are formulated. Twelve boundary and continuity conditions are listed as follows

$$y_1(0) = y_2(2l) = y_3(2l) = y_3(3l) = 0, \quad y_1(l) = y_2(l) = 1, \quad y_1'(0) = 0,$$

$$y_1'(l) = y_2'(l) = 0, \quad y_2'(2l) = y_3'(2l), \quad EIy_2''(2l) = EIy_3''(2l), \quad EIy_3''(3l) = 0$$

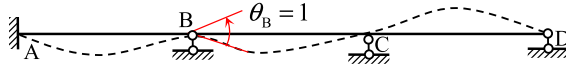
FIG. 7.28 – Influence line shape for F_{RB} of three-span statically indeterminate beam.

As a result, the influence line function for F_{RB} can be obtained as follows:

$$y(x) = \begin{cases} -\frac{2}{l^3}x^3 + \frac{3}{l^2}x^2 & (0 \leq x \leq l) \\ \frac{26}{31l^3}x^3 - \frac{135}{31l^2}x^2 + \frac{192}{31l}x - \frac{52}{31} & (l \leq x \leq 2l) \\ \frac{15}{31l^3}x^3 - \frac{69}{31l^2}x^2 + \frac{60}{31l}x + \frac{36}{31} & (2l \leq x \leq 3l) \end{cases}$$

(2) Construct the influence line for M_B .

The released beam is divided into three segments. Then, impose unit relative rotation $\theta_B = 1$ at the left and right sides of hinge B, and outline the shape of influence line for M_B , as shown in figure 7.29.

FIG. 7.29 – Influence line shape for M_B of three-span statically indeterminate beam.

Combine the deflection equations of released beam with the following 12 boundary and continuous conditions:

$$y_1(0) = y_1(l) = y_2(l) = y_2(2l) = y_3(2l) = y_3(3l) = 0, \quad y'_1(0) = 0,$$

$$y'_1(l) - y'_2(l) = 1, \quad y'_2(2l) = y'_3(2l), \quad EIy''_1(l) = EIy''_2(l),$$

$$EIy''_2(2l) = EIy''_3(2l), \quad EIy''_3(3l) = 0$$

After the undetermined coefficients are addressed, the analytical equation of influence line for M_B is expressed as:

$$y(x) = \begin{cases} \frac{6}{13l^2}x^3 - \frac{6}{13l}x^2 & (0 \leq x \leq l) \\ -\frac{5}{13l^2}x^3 + \frac{27}{13l}x^2 - \frac{46}{13}x + \frac{24l}{13} & (l \leq x \leq 2l) \\ \frac{1}{13l^2}x^3 - \frac{9}{13l}x^2 + 2x - \frac{24l}{13} & (2l \leq x \leq 3l) \end{cases}$$

(3) Construct the influence line for M_E .

The released structure is divided into four segments. Then, impose the unit relative rotation $\theta_E = 1$ at the left and right sides of hinge E, and outline the shape of influence line for M_E , as exhibited in figure 7.30.

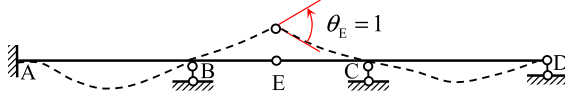


FIG. 7.30 – Shape of influence line for M_E of three-span statically indeterminate beam.

Combining the deflection functions of released structure with 16 boundary and continuous conditions as follows:

$$y_1(0) = y_1(l) = y_2(l) = y_3(2l) = y_4(2l) = y_4(3l) = 0, \quad y_2(3l/2) = y_3(3l/2),$$

$$y'_1(0) = 0, \quad y'_1(l) = y'_2(l), \quad y'_2(3l/2) - y'_3(3l/2) = 1, \quad y'_3(2l) = y'_4(2l),$$

$$EIy''_1(l) = EIy''_2(l), \quad EIy''_2(3l/2) = EIy''_3(3l/2), \quad EIy''_3(2l) = EIy''_4(2l),$$

$$EIy''_4(3l) = 0, \quad EIy''_2(3l/2) = EIy'''_3(3l/2)$$

the influence line function for M_E is obtained as

$$y(x) = \begin{cases} \frac{9}{52l^2}x^3 - \frac{9}{52l}x^2 & (0 \leq x \leq l) \\ -\frac{1}{52l^2}x^3 + \frac{21}{52l}x^2 - \frac{15}{26}x + \frac{5l}{26} & (l \leq x \leq \frac{3}{2}l) \\ -\frac{1}{52l^2}x^3 + \frac{21}{52l}x^2 - \frac{41}{26}x + \frac{22l}{13} & (\frac{3}{2}l \leq x \leq 2l) \\ -\frac{5}{52l^2}x^3 + \frac{45}{52l}x^2 - \frac{5}{2}x + \frac{30l}{13} & (2l \leq x \leq 3l) \end{cases}$$

Finally, take $l = 6$ m and draw the influence lines for M_B and M_E , respectively, as displayed in figure 7.31.

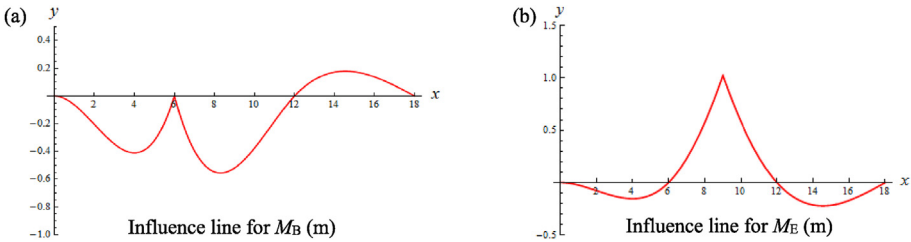


FIG. 7.31 – Influence line for M_B and M_E of three-span statically indeterminate beam.

Problems

7.1 Apply the equilibrium method to draw the influence lines for reactions and internal forces of structures:

- (a) F_{yA} , M_A , M_C , F_{SC} .
- (b) F_{yA} , M_C , F_{SC} , F_{NC} of inclined beam.
- (c) F_{SA}^L , F_{SA}^R , F_{SC} , M_C .
- (d) F_{RA} , F_{SB} , M_E , F_{SF} .

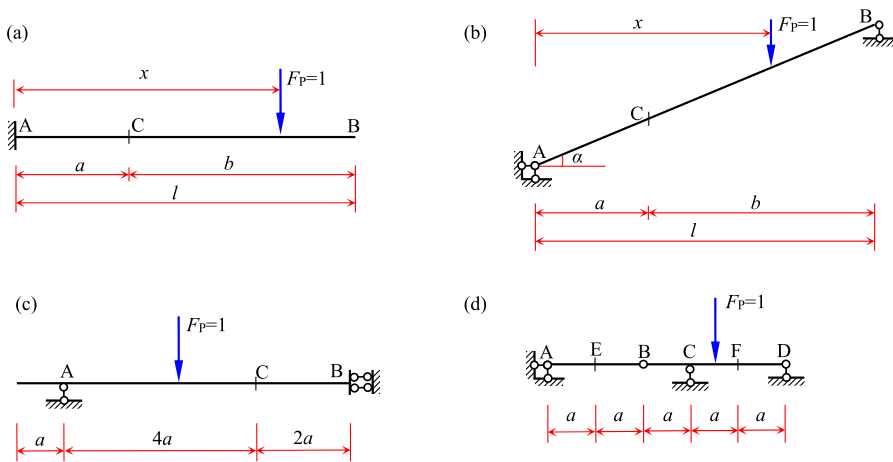


FIG. P7.1

7.2 Apply the equilibrium method to draw the influence lines for F_{RA} , F_{SB} , M_E , F_{SE} , F_{RC} , F_{RD} , M_F , and F_{SF} of beam.

Remark: The influence lines for various responses of auxiliary part (AB) are the same as those of simply supported beam, and there is no ordinate in the basic part (BD).

The influence lines for various responses of basic part (BD) in segment BD are the same as those of the cantilever beam BD, and it is straight line in segment AB.

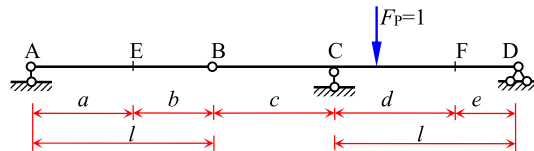


FIG. P7.2

7.3 Apply the equilibrium method to construct the influence lines for M_A , F_{yA} , M_K , and F_{SK} of the frame. Suppose that both M_A and M_K making the inside of member in tension are positive.

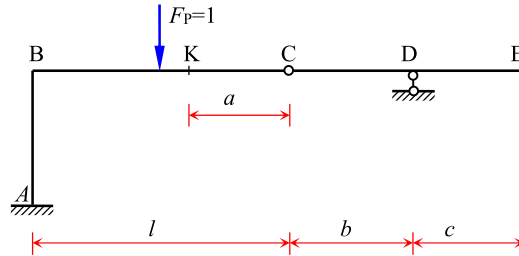


FIG. P7.3

7.4 Apply the equilibrium method to construct the influence lines for F_{RA} , F_{RC} , F_{SB}^L , F_{SB}^R , M_F , F_{SF} , M_G and F_{SG} of the multispan statically determinate beam.

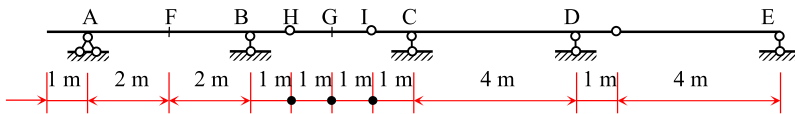


FIG. P7.4

7.5 Apply the equilibrium method to construct the influence lines for F_{RA} , F_{RB} and M_A of the multispan statically determinate beam.

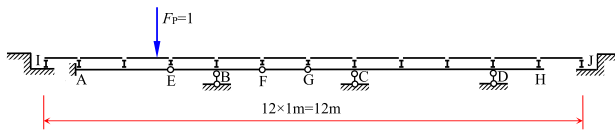


FIG. P7.5

7.6 Apply the equilibrium method to draw the influence lines for axial forces F_{N1} , F_{N2} and F_{N3} of the truss (the loadings are divided into the two cases of acting at the top and bottom cords, respectively).

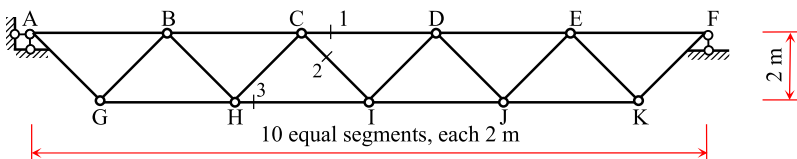


FIG. P7.6

7.7 Construct the influence lines for M_C and F_{SC} of frame, where the unit horizontal load moves along the column height.

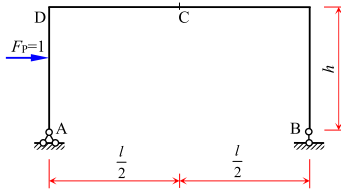


FIG. P7.7

7.8 Apply the kinematic method to draw the influence lines for M_E , F_{SB}^L and F_{SB}^R of beam.

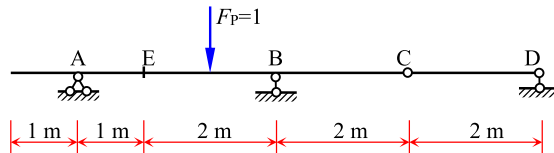


FIG. P7.8

7.9 Apply the kinematic method to redo problems 7.4 and 7.5.

7.10 Apply the kinematic method to draw the influence lines for axial forces F_{N1} , F_{N2} , F_{N3} and F_{N4} of the truss.

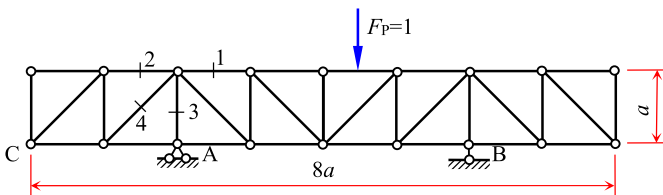


FIG. P7.10

7.11 Determine the reactions F_{RA} , F_{RB} and internal forces F_{SC} and M_C of the beam under the loads by using the influence line.

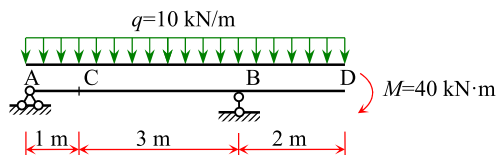


FIG. P7.11

7.12 Determine the most unfavorable position of motorcade on the influence line for response Z shown in the figure and the absolute maximum value of Z .

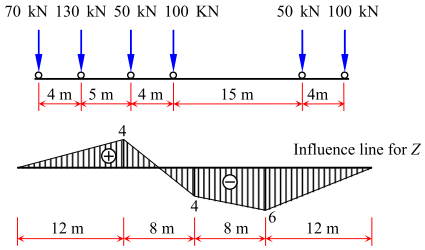


FIG. P7.12

7.13 Determine the most unfavorable position of loads for M_C and F_{SC} of crane beams with the moving two cranes shown in the figure, and calculate the corresponding maximum and minimum responses.

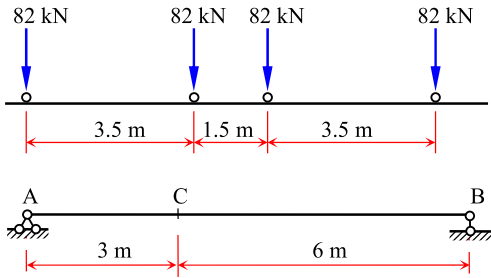


FIG. P7.13

7.14 Construct the influence line for shear F_{SC} of section C in the statically indeterminate beam.

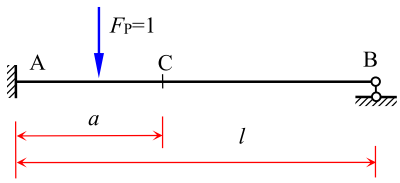


FIG. P7.14

7.15 Construct the influence line for moment M_I of section I at the middle of span BC in the four-span frame with 9 degrees of indeterminacy.

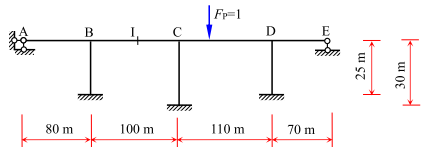


FIG. P7.15

Chapter 8

Matrix Displacement Methods

For the analysis of force and deformation of complex structures, traditional approaches of structural analysis rely heavily on hand calculation in which there may exist serious difficulties. On the one hand, the computational cost by hand is huge. On the other hand, the calculation procedure is too complicated to prevent making mistakes. The shortcomings of traditional approaches have limited their wide application in engineering. With the development of computer and computational science and technology, it is demonstrated that the mechanical analysis approaches based on computer calculation are powerful for reducing the demanding computational efforts. Thus, the engineers can be free from cumbersome manual calculations and focus on other creative works. In conjunction with the blooming of engineering digitalization design, informatization, and intelligentization technology, computer-aided and automatic structural analysis has received extensive attention and made great progress. In fact, the **matrix displacement method** (or, matrix stiffness method) is representative.

By the 1950s, the Greek mechanician Argyris J.H. proposed the matrix analysis method (including matrix displacement method and matrix force method) for the analysis of complex structures in the aircraft industry. Like the traditional force method, the matrix force method requires choosing a primary structure. This approach is rather complicated since many cases should be considered during the implementation, and at present it is rarely adopted. The matrix displacement method establishes the stiffness matrix for each element based on the slope-deflection equation and assembles the matrices of all the elements into a global stiffness matrix. Then, it constructs the equivalent nodal force vector and finally forms the global stiffness equation corresponding to the displacement method. Solving the global equation gives nodal displacements, and the nodal internal forces of the elements can be further obtained. The matrix displacement method has the following features: the numbering of nodal displacements and member elements as well as global structure are organized in a unified way. A standard and modularization procedure is developed for forming the elemental and global stiffness matrices, and the global stiffness equation is solved conveniently through computer programming which matches the requirement of automatic computing. Actually, the matrix displacement method is an early form of the later-developed finite element method for continua. Therefore, it

is also termed the finite element method for member systems, and it can be derived *via* the principle of minimum potential energy. Learning the matrix displacement method is helpful for students to understand some important concepts and computational procedures of the finite element method for continuum structures, and is a good way to deepen their comprehension on the finite element analysis.

This chapter introduces the fundamental principle and some details of the matrix displacement method, including the elemental stiffness matrix and coordinate transformation for bending member, continuous beam and axial force member, global stiffness matrix of structure, equivalent nodal loads, computational procedure, and illustrative examples.

8.1 Fundamental Principle of the Matrix Displacement Method

The matrix displacement method is an approach of matrix analysis for member structures, which is based on the displacement method. In other words, it is a variant of the displacement method which is expressed in the form of a matrix and is more suitable to be solved with a computer. The nodal displacements in the matrix displacement method are also taken as primary unknowns. In this way, the compatibility condition of deformation has already been satisfied when choosing the primary unknowns. According to the idea of the displacement method for selecting the primary system, the corresponding primary structure is unique. Slightly different from the displacement method, the matrix displacement method generally considers the axial deformation of frame members and treats all the members of the frame including statically determinate elements as the beams with two ends clamped. Hence, only one type of basic member with two ends clamped is defined in the matrix displacement method. In this way, the number of primary unknowns can be determined easily, and it is convenient to standardize the procedures of structural analysis and calculation.

The matrix displacement method consists of two steps: the first one is **elemental analysis** and the second one is **global analysis**. In a member structure, a member is normally taken as an element, and thus the whole structure is divided into many elements. It is seen that the member structures are naturally discrete structures. The purpose of elemental analysis is to construct the elemental stiffness equation and form the elemental stiffness matrix. The elemental stiffness equation refers to the equation that the end force vector $\bar{\mathbf{F}}^e$ and the end displacement vector $\bar{\mathbf{\Delta}}^e$ of a member element e satisfy, and it is expressed as follows:

$$\bar{\mathbf{F}}^e = \bar{\mathbf{k}}^e \bar{\mathbf{\Delta}}^e$$

where $\bar{\mathbf{k}}^e$ is the elemental stiffness matrix of element e .

The purpose of global analysis is to combine all the elements together, to form the global stiffness matrix by assembling all the element stiffness matrices following certain rules, to obtain the basic equation of the matrix displacement method, and

to solve the nodal displacements and inner forces. Similarly, the global stiffness matrix is the one the nodal force vector $\mathbf{F}$ and nodal displacement vector $\mathbf{A}$ have to satisfy: $\mathbf{F} = \mathbf{K}\mathbf{A}$. Furthermore, the basic equation of the matrix displacement method in the form of a matrix is specified by

$$\mathbf{K}\mathbf{A} = \mathbf{P}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{A}$ includes all the unknown nodal displacements and $\mathbf{P}$ includes all the nodal forces related to the unknown nodal displacements, the non-nodal loads should normally be transformed into equivalent nodal forces. The assembling rule to form the global stiffness matrix from all the elemental stiffness matrices is a core content of the matrix displacement method.

Then, solving the basic equation of the matrix displacement method, which is a set of linear algebraic equations, yields the nodal displacements:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{K}$$

Once all the nodal displacements are solved, the nodal forces and the reaction forces from supports can be calculated directly with the elemental stiffness equation, and the internal forces of the structure can be further obtained. The main advantage of the matrix displacement method is the uniform and standard solving procedure, *i.e.*, all the elements are treated in the same way during the analysis, and this feature is definitely beneficial for computer programming implementation since it makes full use of the advantage of computer in dealing with repetitive works to conduct structural mechanics analysis with high efficiency.

8.2 Elemental Stiffness Matrix

An element is a single member of a member structure. The elemental stiffness matrix reflects the relationship between the member end forces and the member end displacements for two ends of an element. Figure 8.1 depicts an element e of a straight member with an equal cross-section in the planar frame, in which the length of the member is l , the cross-section area is A , the moment of inertia is I and elastic modulus is E . For generality and also accurate description, the effect of axial deformation is considered in addition to the bending deformation of the element. There are 3 displacement components on each end including two line displacements and one rotation, so in the most general case, there are a total of 6 displacement components for an element of the plane member system.

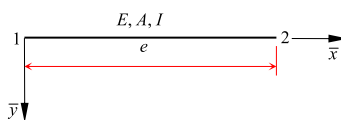


FIG. 8.1 – Element of a straight member in planar frame.

The local numbering as “1” and “2” of two endpoints (nodes) of an element is adopted, as shown in figure 8.1. The direction from end 1 toward end 2 is specified as the positive direction of the member axis. In the figure, the coordinate system $\bar{x}\bar{y}$ is adopted where the axis $\bar{x}$ overlaps the member axis. The coordinate system is called the **local coordinate system**. The bar above the letters x and y , i.e., $\bar{x}$ and $\bar{y}$, is the mark of the local coordinate system.

8.2.1 Elemental Stiffness Matrix in Local Coordinate System

In the local coordinate system, each end of a general element has normally 3 displacement components $\bar{u}$, $\bar{v}$, and $\bar{\theta}$, and the corresponding force components $\bar{F}_x$, $\bar{F}_y$, $\bar{M}$, respectively. The positive directions of displacement and force components are demonstrated in figure 8.2.

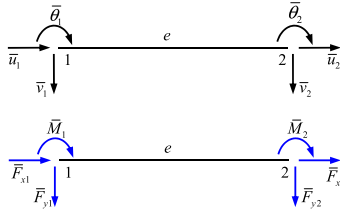


FIG. 8.2 – Positive directions of displacement and force components of an element.

The member end displacement vector $\bar{\mathbf{A}}^e$ and the end force vector $\bar{\mathbf{F}}^e$ of element are formed by arranging the 6 member end displacement components and the 6 end force components in a certain sequence:

$$\left. \begin{aligned} \bar{\mathbf{A}}^e &= (\bar{A}_{(1)} \bar{A}_{(2)} \bar{A}_{(3)} \bar{A}_{(4)} \bar{A}_{(5)} \bar{A}_{(6)})^{eT} = (\bar{u}_1 \bar{v}_1 \bar{\theta}_1 \bar{u}_2 \bar{v}_2 \bar{\theta}_2)^{eT} \\ \bar{\mathbf{F}}^e &= (\bar{F}_{(1)} \bar{F}_{(2)} \bar{F}_{(3)} \bar{F}_{(4)} \bar{F}_{(5)} \bar{F}_{(6)})^{eT} = (\bar{F}_{x1} \bar{F}_{y1} \bar{M}_1 \bar{F}_{x2} \bar{F}_{y2} \bar{M}_2)^{eT} \end{aligned} \right\} \quad (8.1)$$

The 6 members in the above vectors are numbered (1), (2), ..., (6). Since they are numbered in each element separately and are not numbered uniformly in all elements of the frame, they are also called local numbering. The local numbers (1), (2), ... of the member end displacement components (or member end force components) are marked in brackets, which is a sign of local numbering.

Now we discuss the elemental stiffness equation. The **elemental stiffness equation** is the equation that is established when solving the member end forces with member end displacements, it is also denoted as the “ $\bar{\mathbf{A}} \rightarrow \bar{\mathbf{F}}$ ” equation. For small deformation problems, the coupling between the deformations from axial loads and bending loads is negligible, so the elemental stiffness equations for each case are derived separately.

Firstly, the member end forces $\bar{F}_{x1}$ and $\bar{F}_{x2}$ can be calculated with the member end displacements $\bar{u}_1$ and $\bar{u}_2$:

$$\left. \begin{aligned} \bar{F}_{x1}^e &= \frac{EA}{l} (\bar{u}_1^e - \bar{u}_2^e) \\ \bar{F}_{x2}^e &= -\frac{EA}{l} (\bar{u}_1^e - \bar{u}_2^e) \end{aligned} \right\} \quad (8.2)$$

Secondly, the member end transverse forces $\bar{F}_{y1}$, $\bar{F}_{y2}$ and the member end moments $\bar{M}_1$, $\bar{M}_2$ can be calculated with the member end transverse displacements $\bar{v}_1$, $\bar{v}_2$ and end rotations $\bar{\theta}_1$, $\bar{\theta}_2$. According to the slope deflection equations (6.5) and (6.6), using the notation as well as plus and minus sign specification in this chapter yields

$$\left. \begin{aligned} \bar{M}_1^e &= \frac{4EI}{l} \bar{\theta}_1^e + \frac{2EI}{l} \bar{\theta}_2^e + \frac{6EI}{l^2} (\bar{v}_1^e - \bar{v}_2^e) \\ \bar{M}_2^e &= \frac{2EI}{l} \bar{\theta}_1^e + \frac{4EI}{l} \bar{\theta}_2^e + \frac{6EI}{l^2} (\bar{v}_1^e - \bar{v}_2^e) \\ \bar{F}_{y1}^e &= \frac{6EI}{l^2} (\bar{\theta}_1^e + \bar{\theta}_2^e) + \frac{12EI}{l^3} (\bar{v}_1^e - \bar{v}_2^e) \\ \bar{F}_{y2}^e &= -\frac{6EI}{l^2} (\bar{\theta}_1^e + \bar{\theta}_2^e) - \frac{12EI}{l^3} (\bar{v}_1^e - \bar{v}_2^e) \end{aligned} \right\} \quad (8.3)$$

Actually, the 6 stiffness equations in equations (8.2) and (8.3) have been derived in the chapter of displacement method. Now, we combine and rewrite them in form of matrix, namely

$$\begin{bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{M}_1 \\ \text{---} \\ \bar{F}_{x2} \\ \bar{F}_{y2} \\ \bar{M}_2 \end{bmatrix}^e = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \text{---} \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{bmatrix}^e \quad (8.4)$$

Equation (8.4) is rewritten as

$$\bar{\mathbf{F}}^e = \bar{\mathbf{k}}^e \bar{\mathbf{A}}^e \quad (8.5)$$

where

$$\bar{\mathbf{k}}^e = \left[\begin{array}{ccc|ccc} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ \hline -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{array} \right]^e \quad (8.6)$$

Equation (8.5) is the elemental stiffness equation in local coordinate system, and $\bar{\mathbf{k}}^e$ is the **elemental stiffness matrix** in the local coordinate system which is a 6×6 square matrix.

8.2.2 Properties of Elemental Stiffness Matrix

(1) Physical meaning of elemental stiffness coefficients

The elements in $\bar{\mathbf{k}}^e$ are termed as the elemental stiffness coefficients, which represent the member end forces caused by unit displacement of member end. Normally, the coefficient $\bar{k}_{ij}^e$ in the i th row and the j th column indicates the value of the i th end force component $\bar{F}_i^e$ when the j th end displacement component $\bar{A}_j^e$ equals 1 (while the other displacement components are zeroes). In simple words, the first subscript of an elemental stiffness coefficient represents the location of member end force, whereas the second subscript represents the reason for resulting in this member end force.

(2) Symmetry

The symmetry of $\bar{\mathbf{k}}^e$ refers to the elements in it satisfy the following relationship:

$$\bar{k}_{ij}^e = \bar{k}_{ji}^e \quad (8.7)$$

In fact, this property also is easy to be derived from the theorem of reciprocal reactions.

(3) Singularity

The stiffness matrix $\bar{\mathbf{k}}^e$ of a general element is a singular matrix. If one adds the 1st row (or column) with the 4th row (or column), the resultant coefficients of one row (column) all equal 0. Similarly, adding the 2nd row (column) with the 5th row

(column) also equal 0. It indicates that the determinant of the matrix $\bar{\mathbf{k}}^e$ is zero, *i.e.*,

$$|\bar{\mathbf{k}}^e| = 0 \quad (8.8)$$

Thus, the matrix is singular and cannot be inverted. Hence, the member end force $\bar{\mathbf{F}}^e$ can be determined with a specified end displacement $\bar{\mathbf{A}}^e$ according to equation (8.4). Nevertheless, one cannot solve the member end displacement $\bar{\mathbf{A}}^e$ with a specified end force $\bar{\mathbf{F}}^e$ using the same equation. This point can be explained by the concepts of mechanics. Since the discussed is a general element, of which the two ends are free from any support constraint, the member contains any rigid body displacement apart from the axial and bending deformations resulting from the end forces. Hence, the specific member end force $\bar{\mathbf{F}}^e$ cannot yield a unique solution of $\bar{\mathbf{A}}^e$, unless sufficient displacement constraint conditions are applied.

Finally, it is pointed out that, equation (8.4) is the stiffness equation of a general element, where 6 member end displacements can be set arbitrary values. There are some special elements in structures, namely, one or some member end displacements of the element are known as 0, which cannot be set arbitrarily. The stiffness equations of special elements need not to re-derived additionally, which can be obtained automatically with some special treatments on the stiffness equation (8.4) of a general element. The structural matrix analysis focuses on the programming, standardization, automation of computational procedures. Therefore, the stiffness matrices of non-standard special elements are not presented usually to avoid unnecessary complex operations. The standard form, *i.e.*, stiffness matrix (8.6) of a general element is adopted, and various special forms of elemental stiffness matrix will be formed automatically by computer programming.

8.3 Coordinate Transformation of Elemental Stiffness Matrix

8.3.1 Elemental Stiffness Matrix in the Global Coordinate System

In the previous section, the elemental stiffness matrix in the local coordinate system is established. Note that, the directions of the member axes of a complex structure are usually distinct, and the local coordinate systems of these members are different and are not uniform. For the convenience of structural analysis, one has to choose a public coordinate system, which is also termed the **global coordinate system** or structural coordinate system. In what follows, the elemental stiffness matrix in the local coordinate system is transformed into the global coordinate system by means of coordinate transformation.

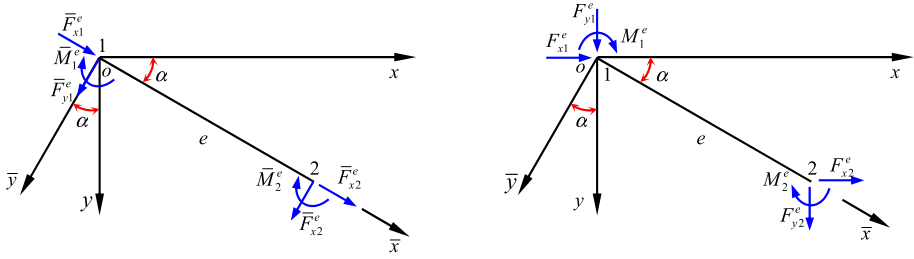


FIG. 8.3 – Relation between local and global coordinate systems.

Firstly, a global coordinate system Oxy is defined, and the angle α between its x -axis and the axial axis $\bar{x}$ of the local coordinate system is considered to be positive when it is along the clockwise direction, as shown in figure 8.3. The member end force components in the local coordinate system are denoted by $\bar{F}_x^e$, $\bar{F}_y^e$, $\bar{M}^e$ as shown in figure 8.3a, and the member end force components in the global coordinate system are denoted by F_x^e , F_y^e , M^e as shown in figure 8.3b. Obviously, the nodal force components in the local coordinate system can be expressed in terms of the nodal forces in the global coordinate system, and the relationship between them is specified by

$$\left. \begin{aligned} \bar{F}_{x1}^e &= F_{x1}^e \cos \alpha + F_{y1}^e \sin \alpha \\ \bar{F}_{y1}^e &= -F_{x1}^e \sin \alpha + F_{y1}^e \cos \alpha \\ \bar{M}_{x1}^e &= M_{x1}^e \\ \bar{F}_{x2}^e &= F_{x2}^e \cos \alpha + F_{y2}^e \sin \alpha \\ \bar{F}_{y2}^e &= -F_{x2}^e \sin \alpha + F_{y2}^e \cos \alpha \\ \bar{M}_2^e &= M_2^e \end{aligned} \right\} \quad (8.9)$$

Rewriting the above equation in the form of matrix gives

$$\begin{bmatrix} \bar{F}_{x1}^e \\ \bar{F}_{y1}^e \\ \bar{M}_1^e \\ \bar{F}_{x2}^e \\ \bar{F}_{y2}^e \\ \bar{M}_2^e \end{bmatrix}^e = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{x1}^e \\ F_{y1}^e \\ M_1^e \\ F_{x2}^e \\ F_{y2}^e \\ M_2^e \end{bmatrix}^e \quad (8.10)$$

Equation (8.10) can be expressed in an abbreviated form as

$$\bar{\mathbf{F}}^e = \mathbf{T} \mathbf{F}^e \quad (8.11)$$

where the matrix $\mathbf{T}$ is termed as the coordinate transformation matrix of an element, and it is specified by

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \quad (8.12)$$

In fact, equation (8.11) is the transformation formulation for the member end forces of the element in two coordinate systems.

It can be proven that $\mathbf{T}$ is an orthotropic matrix. The inverse of $\mathbf{T}$ is equal to its transpose, *i.e.*,

$$\mathbf{T}^{-1} = \mathbf{T}^T \quad (8.13)$$

Inversing equation (8.11) gives

$$\mathbf{F}^e = \mathbf{T}^T \bar{\mathbf{F}}^e \quad (8.14)$$

Likewise, the transformation relation of end displacements between the two coordinate systems can be determined. Denote the end displacement vectors in the local and global coordinate systems as $\bar{\mathbf{A}}^e$ and $\mathbf{A}^e$, respectively. One has

$$\bar{\mathbf{A}}^e = \mathbf{T} \mathbf{A}^e \quad (8.15)$$

$$\mathbf{A}^e = \mathbf{T}^T \bar{\mathbf{A}}^e \quad (8.16)$$

The coordinate transformation between element end force and end displacement can be expressed in the form of

$$\mathbf{F}^e = \mathbf{k}^e \mathbf{A}^e \quad (8.17)$$

where $\mathbf{k}^e$ is the elemental stiffness matrix in the global coordinate system.

In the following, we start to perform the coordinate transformation between the elemental stiffness matrix $\mathbf{k}^e$ under global coordinate system and $\bar{\mathbf{k}}^e$ under local coordinate system.

The elemental stiffness equation under the local coordinate system for an element e is given by

$$\bar{\mathbf{F}}^e = \bar{\mathbf{k}}^e \bar{\mathbf{A}}^e \quad (\text{a})$$

Substituting equations (8.11) and (8.15) in equation (a) gives

$$\mathbf{T} \mathbf{F}^e = \bar{\mathbf{k}}^e \mathbf{T} \mathbf{A}^e$$

Multiplying both sides of the above equation with $\mathbf{T}^T$ gives

$$\mathbf{F}^e = \mathbf{T}^T \bar{\mathbf{k}}^e \mathbf{T} \mathbf{A}^e \quad (\text{b})$$

If one compares equation (b) and (8.17), it is found that

$$\mathbf{F}^e = \mathbf{T}^T \bar{\mathbf{k}}^e \mathbf{T} \mathbf{\Delta}^e \quad (8.18)$$

Equation (8.18) is the coordinate transformation of an elemental stiffness matrix between the two coordinate systems. The coordinate transformation matrix, $\mathbf{k}^e$ can be calculated with $\bar{\mathbf{k}}^e$ through equation (8.18).

The elemental stiffness matrix $\mathbf{k}^e$ has the same size with $\bar{\mathbf{k}}^e$, and it also has similar properties:

- (1) The member k_{ij}^e represents the value of the i th nodal force component when the j th nodal displacement component is equal to 1.
- (2) $\mathbf{k}^e$ is a symmetric matrix.
- (3) $\mathbf{k}^e$ is a singular matrix.

Example 8.1. Determine the elemental stiffness matrix $\mathbf{k}^e$ of all the members of the frame structure as shown in figure 8.4. The geometric dimensions are specified by: length $l = 5$ m, cross-section size $bh = 0.5 \text{ m} \times 1 \text{ m}$, $A = 0.5 \text{ m}^2$, $E = 3 \times 10^4$ MPa, $EA/l = 300 \times 10^4 \text{ kN/m}$, $I = 1/24 \text{ m}^4$, $EI/l = 25 \times 10^4 \text{ kN/m}$.

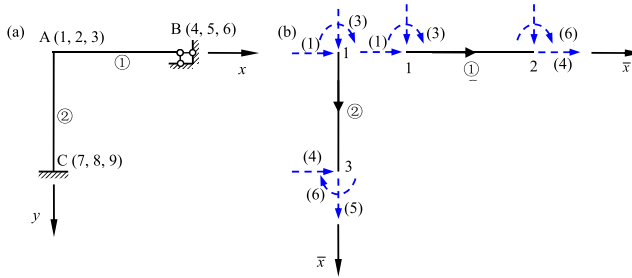


FIG. 8.4 – Elements of a frame structure.

Solution: (1) Calculate the elemental stiffness matrix in the local coordinate system $\bar{\mathbf{k}}^e$.

Figure 8.4a is the global coordinate system while figure 8.4b is the local coordinate system. There are 2 elements ① and ② in the frame which link points A and B, and A and C, respectively. Points A, B, and C are identified as nodes 1, 2 and 3, respectively.

Because elements ① and ② have the same size, so $\bar{\mathbf{k}}^{①}$ and $\bar{\mathbf{k}}^{②}$ should be equal to each other. According to equation (8.6), it gives

$$\bar{\mathbf{k}}^{\textcircled{1}} = \bar{\mathbf{k}}^{\textcircled{2}} = 10^4 \times \left[\begin{array}{ccc|ccc} 300 \text{ kN/m} & 0 & 0 & -300 \text{ kN/m} & 0 & 0 \\ 0 & 12 \text{ kN/m} & 30 \text{ kN} & 0 & -12 \text{ kN/m} & 30 \text{ kN} \\ 0 & 30 \text{ kN} & 100 \text{ kN}\cdot\text{m} & 0 & -30 \text{ kN} & 50 \text{ kN}\cdot\text{m} \\ \hline -300 \text{ kN/m} & 0 & 0 & 300 \text{ kN/m} & 0 & 0 \\ 0 & -12 \text{ kN/m} & -30 \text{ kN} & 0 & 12 \text{ kN/m} & -30 \text{ kN} \\ 0 & 30 \text{ kN} & 50 \text{ kN}\cdot\text{m} & 0 & -30 \text{ kN} & 100 \text{ kN}\cdot\text{m} \end{array} \right]$$

(2) Calculate elemental stiffness matrix in the global coordinate system $\mathbf{k}^e$.

For element ①: $\alpha = 0$, $\mathbf{T} = \mathbf{I}$, according to equation (8.18), it gives

$$\mathbf{k}^{\textcircled{1}} = \bar{\mathbf{k}}^{\textcircled{1}} = 10^4 \times \left[\begin{array}{ccc|ccc} 300 & 0 & 0 & -300 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ \hline -300 & 0 & 0 & 300 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{array} \right] \quad (8.19)$$

For element ②: $\alpha = 90^\circ$, the coordinate transformation matrix is specified by

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 0 \\ & \mathbf{0} & & -1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array} \right]$$

The elemental stiffness matrix in the global coordinate system is specified by

$$\mathbf{k}^{\textcircled{2}} = \mathbf{T}^T \bar{\mathbf{k}}^{\textcircled{2}} \mathbf{T} = 10^4 \times \left[\begin{array}{ccc|ccc} 12 & 0 & -30 & -12 & 0 & -30 \\ 0 & 300 & 0 & 0 & -300 & 0 \\ -30 & 0 & 100 & 30 & 0 & 50 \\ \hline -12 & 0 & 30 & 12 & 0 & 30 \\ 0 & -300 & 0 & 0 & 300 & 0 \\ -30 & 0 & 50 & 30 & 0 & 100 \end{array} \right] \quad (8.20)$$

8.3.2 Elemental Stiffness Matrix of Continuous Beam

For a general element, the 6 nodal displacements are all primarily unknown, so all the elements in the elemental stiffness matrix must be considered. In fact, there are a few special structures that exist in the structural analysis, in which some nodal displacement components of elements are known. Therefore, the known

displacements need to be taken into account in the elemental stiffness matrix, and some coefficients should be deleted to obtain a reduced elemental stiffness matrix. The continuous beam structure as shown in figure 8.5 is a typical case, in which the axial deformation is normally neglected for structural analysis, and the vertical displacements of the 3 nodes are all zero. Thus, only the rotational angles of 2 nodes in each element are unknown, and the corresponding nodal forces are member end moments.

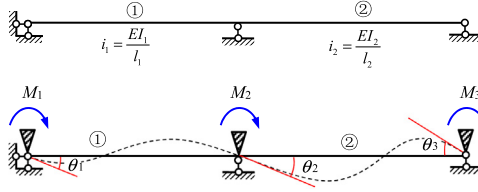


FIG. 8.5 – Nodal displacement unknowns of elements in continuous beam.

According to equation (8.3), in case only the rotations at the nodes of the element are unknown the relation between nodal moment and rotation is formulated as

$$\left. \begin{aligned} \overline{M}_1^e &= \frac{4EI}{l} \overline{\theta}_1^e + \frac{2EI}{l} \overline{\theta}_2^e \\ \overline{M}_2^e &= \frac{2EI}{l} \overline{\theta}_1^e + \frac{4EI}{l} \overline{\theta}_2^e \end{aligned} \right\} \quad (8.21)$$

From equation (8.21), the elemental stiffness matrix of the continuous beam in the local coordinate system is

$$\overline{\mathbf{k}}^e = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \quad (8.22)$$

It is seen that the elemental stiffness matrix of the continuous beam can be obtained by deleting the 1st, 2nd, 4th, and 5th rows and columns from the general elemental stiffness matrix. Herein, an extended conclusion is obtained: when one nodal displacement of an element is known as zero, the corresponding row and column should be deleted from the general elemental stiffness matrix to achieve the stiffness matrix of this element.

Since the bending moments under the global and local coordinates are equal to each other, *i.e.*,

$$\begin{cases} \overline{M}_1^e = M_1^e \\ \overline{M}_2^e = M_2^e \end{cases} \quad (c)$$

Rewriting equation (c) in the form of matrix gives

$$\begin{bmatrix} \overline{M}_1^e \\ \overline{M}_2^e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1^e \\ M_2^e \end{bmatrix} \quad (8.23)$$

It can be simplified into

$$\overline{\mathbf{F}}^e = \mathbf{T} \mathbf{F}^e \quad (8.24)$$

where the transformation matrix $\mathbf{T}$ equals the identity matrix $\mathbf{I}$.

Similarly, the element nodal displacements $\overline{\mathbf{A}}^e$ of continuous beam in local coordinate system can also be represented by using the element nodal displacements $\mathbf{A}^e$ in global coordinate system

$$\overline{\mathbf{A}}^e = \mathbf{I} \mathbf{A}^e \quad (8.25)$$

On the contrary, the element nodal forces and displacements under the global coordinate system can be represented by those under the local coordinate system, namely

$$\mathbf{F}^e = \mathbf{I} \overline{\mathbf{F}}^e \quad (d)$$

$$\mathbf{A}^e = \mathbf{I} \overline{\mathbf{A}}^e \quad (e)$$

Consider the elemental stiffness equation in the local coordinate system, and substitute the coordinate transformation matrix into it, the following equation can be obtained

$$\mathbf{I} \mathbf{F}^e = \overline{\mathbf{k}}^e \mathbf{I} \mathbf{A}^e \quad (f)$$

The elemental stiffness matrix in the global coordinate system of the continuous beam can be further obtained

$$\mathbf{F}^e = \mathbf{I}^T \overline{\mathbf{k}}^e \mathbf{T} \mathbf{A}^e = \mathbf{k}^e \mathbf{A}^e \quad (8.26)$$

Thus

$$\mathbf{k}^e = \mathbf{I}^T \overline{\mathbf{k}}^e \mathbf{I} = \overline{\mathbf{k}}^e \quad (8.27)$$

It is found that the elemental stiffness matrices of a continuous beam in the local and global coordinate systems are identical.

8.3.3 Elemental Stiffness Matrix of Axial Force Bar

Besides continuous beams, trusses are also widely used structures in engineering. The specialness lies in that, the members in a truss structure are connected through smooth pins, and the directions of resultant forces at the member end coincide with the bar axis. As a result, the bar is always under axial tension or axial compression state. For an axial bar element of a truss, the nodal forces are only related to end displacements

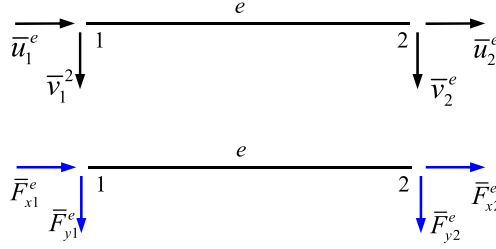


FIG. 8.6 – Nodal displacements and forces of the axial force barelement in a truss.

and are independent of nodal rotations. Hence, the elements in the stiffness matrix related to nodal rotations should be eliminated in the analysis. For the axial force bar element in the truss as shown in figure 8.6, the relationship between nodal forces and nodal displacements under the local coordinate system is expressed by

$$\left. \begin{aligned} \bar{F}_{x1}^e &= \frac{EA}{l} (\bar{u}_1^e - \bar{u}_2^e) \\ \bar{F}_{x2}^e &= -\frac{EA}{l} (\bar{u}_1^e - \bar{u}_2^e) \\ \bar{F}_{y1}^e &= 0 \\ \bar{F}_{y2}^e &= 0 \end{aligned} \right\} \quad (8.28)$$

The matrix form of the above equations is

$$\begin{bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{F}_{x2} \\ \bar{F}_{y2} \end{bmatrix}^e = \begin{bmatrix} \frac{EA}{l} & 0 & -\frac{EA}{l} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{l} & 0 & \frac{EA}{l} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{bmatrix}^e \quad (8.29)$$

From equation (8.29), it is observed that the elemental stiffness matrix is

$$\bar{\mathbf{k}}^e = \begin{bmatrix} \frac{EA}{l} & 0 & -\frac{EA}{l} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{l} & 0 & \frac{EA}{l} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (8.30)$$

The relation between element nodal forces in the local and global coordinate systems is written as

$$\begin{cases} \bar{F}_{x1}^e = F_{x1}^e \cos \alpha + F_{y1}^e \sin \alpha \\ \bar{F}_{y1}^e = -F_{x1}^e \sin \alpha + F_{y1}^e \cos \alpha \\ \bar{F}_{x2}^e = F_{x2}^e \cos \alpha + F_{y2}^e \sin \alpha \\ \bar{F}_{y2}^e = -F_{x2}^e \sin \alpha + F_{y2}^e \cos \alpha \end{cases} \quad (g)$$

Rewriting equation (g) in the form of matrix yields

$$\begin{bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ - \\ \bar{F}_{x2} \\ \bar{F}_{y2} \end{bmatrix}^e = \begin{bmatrix} \cos \alpha & \sin \alpha & | & 0 & 0 \\ -\sin \alpha & \cos \alpha & | & 0 & 0 \\ \hline 0 & 0 & | & \cos \alpha & \sin \alpha \\ 0 & 0 & | & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} F_{x1} \\ F_{y1} \\ - \\ F_{x2} \\ F_{y2} \end{bmatrix}^e \quad (8.31)$$

Or equation (8.31) is simplified as

$$\bar{\mathbf{F}}^e = \mathbf{T} \mathbf{F}^e \quad (8.32)$$

where the coordinate transformation matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} \cos \alpha & \sin \alpha & | & 0 & 0 \\ -\sin \alpha & \cos \alpha & | & 0 & 0 \\ \hline 0 & 0 & | & \cos \alpha & \sin \alpha \\ 0 & 0 & | & -\sin \alpha & \cos \alpha \end{bmatrix} \quad (8.33)$$

$\mathbf{F}^e$ is the nodal force vector of axial force bar element in global coordinate system, and it is given by

$$\mathbf{F}^e = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix}^2$$

Similarly, the element nodal displacements $\bar{\mathbf{A}}^e$ under the local coordinate system can also be represented with the counterparts $\mathbf{A}^e$ in the global coordinate system

$$\bar{\mathbf{A}}^e = \mathbf{T} \mathbf{A}^e \quad (h)$$

where $\mathbf{A}^e$ is the nodal displacement vector of axial force bar element in global coordinate system

$$\mathbf{A}^e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}^e$$

On the contrary, the element nodal forces and displacements in global coordinate system are expressed by those in local coordinate system, *i.e.*,

$$\mathbf{F}^e = \mathbf{T}^T \bar{\mathbf{F}}^e \quad (\text{i})$$

$$\mathbf{A}^e = \mathbf{T}^T \bar{\mathbf{A}}^e \quad (\text{j})$$

Considering the elemental stiffness equation of axial force bar under the local coordinate system combined with the coordinate transformation, the following formulation is obtained

$$\mathbf{T}\mathbf{F}^e = \bar{\mathbf{k}}^e \mathbf{T}\mathbf{A}^e$$

In terms of the property of the coordinate transformation matrix $\mathbf{T}$, the elemental stiffness equation under the global coordinate system can be further written as

$$\mathbf{F}^e = \mathbf{T}^T \bar{\mathbf{k}}^e \mathbf{T} \mathbf{A}^e = \mathbf{k}^e \mathbf{A}^e \quad (8.34)$$

where

$$\mathbf{k}^e = \mathbf{T}^T \bar{\mathbf{k}}^e \mathbf{T} \quad (8.35)$$

$\mathbf{k}^e$ is the elemental stiffness matrix of axial force bar in global coordinate system.

8.4 Global Stiffness Matrix of Structure

Section 8.3 discusses the relationship (elemental stiffness matrix) between nodal forces and nodal displacements of an element. On that basis, the relationship between nodal displacements and nodal forces of the whole structure (namely, the **global stiffness matrix**) is further discussed in this section. Global analysis is conducted based on elemental analysis, it derives a global stiffness matrix. Moreover, the geometric and equilibrium conditions on the nodes are considered, a global stiffness equation with the primary unknowns is established, and the basic equation of the matrix displacement method is obtained. Specifically, a frame structure is taken as an example in this section to elaborate the direct stiffness method for forming the global stiffness matrix from elemental stiffness matrices, this method is also termed as the element assembling method or stiffness assembling method.

8.4.1 Element and Node Numberings

In the global analysis of a structure, the element numbers should be defined for the identification of different elements. In the meantime, it is also necessary to define the global numbering of the nodal displacement to determine uniquely the location of one component of node displacement. For the frame shown in figure 8.4, there are 2 elements *i.e.*, element 1 and element 2, denoted by ① and ②, respectively. The 3 nodes are denoted by the unique numberings 1, 2, and 3 (corresponding to nodes A, B, and C). Moreover, the two nodes of element ① are 1 and 2 separately, and the nodes of element ② are 1 and 3. For each node, there are 3 displacement components: u , v and θ , and it makes the frame have a total of 9 displacement components: (1), (2), ..., (9). According to the property of the elemental stiffness matrix, the original global stiffness matrix can be expressed as

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & \dots \\ K_{21} & K_{22} & K_{23} & K_{24} & \dots \\ K_{31} & K_{32} & K_{33} & K_{34} & \dots \\ K_{41} & K_{42} & K_{43} & K_{44} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ \dots \\ \dots \end{bmatrix} \tag{8.36}$$

In the above matrix and vectors, the subscript of the nodal forces and nodal displacements are related to nodal numberings. It can be seen from the subscripts $i = 1, 2, \dots$ that the information of a single node occupies 3 positions in the nodal displacement and force vectors. The locations of the 3 displacements u_i, v_i and θ_i for the i th node in the vectors are $3(i - 1) + 1, 3(i - 1) + 2$ and $3(i - 1) + 3$, respectively. For a structure with N nodes, there should be $3N$ members in the nodal force and displacement vectors, and the size of the original global stiffness matrix should be $3N \times 3N$. Table 8.1 lists the local numbering and the corresponding global numbering of nodal displacements of the 2 elements. It is clearly shown that the local numbering of nodal displacements for each element is identical, in which its order is from (1) to (6). However, the global numbering of nodal displacements should be determined in terms of specific cases. According to the global numbering of nodal displacements in figure 8.4 (the numbers in brackets in figure 8.4a), it is found that the global numbering and local numbering are just the same for element ①, but they are different for element ②. The vector composed of the global numbering of the nodal displacements of an element is called an **elemental locating vector**. It indicates the corresponding relationship between the local and global numbering of nodal displacements of an element, from which the coefficients in the elemental stiffness matrix can be located in the global stiffness matrix in terms of the respective numbering. The locating vectors for elements ① and ② are given by

$$\xi^{①} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \qquad \xi^{②} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

TAB. 8.1 – Local numbering and global numbering of nodal displacements of two elements.

Element ①		Element ②	
Local numbering	Global numbering	Local numbering	Global numbering
(1)	1	(1)	1
(2)	2	(2)	2
(3)	3	(3)	3
(4)	4	(4)	7
(5)	5	(5)	8
(6)	6	(6)	9

8.4.2 Direct Stiffness Method for Assembling Global Stiffness Matrix

The physical meaning of the element of a global stiffness matrix is the nodal force resulting from a unit nodal displacement, the element at the i th row and j th column represents the i th nodal force resulting when the j th nodal displacement is 1 (all the other nodal displacements are 0). In the global analysis, a nodal force should be contributed from all the related elements, so one should sum all the related coefficients in each elemental stiffness matrix, and this process is actually the formation of the original global stiffness matrix through the superposition of the elemental stiffness matrices, and this method is also termed as **direct stiffness method**. The coefficient in an elemental stiffness matrix represents the nodal force resulting from some specified nodal displacement, while one should find out the new positions of the nodal displacement and nodal force under the global numbering in order to identify the position in the global stiffness matrix. In fact, the process of finding the new position in the global stiffness matrix can be realized by using the elemental locating vector. For an element coefficient k_{ij}^e , its position in the global stiffness matrix during assembling the elemental stiffness matrices should be re-examined. Its original row and column numbers are, respectively, i and j , and then the new row and column numbers of the element in the global stiffness matrix are $\xi^e(i)$ and $\xi^e(j)$. It is seen that there are two key steps of locating and accumulation in the direct stiffness method. Through the two steps, the coefficients of elemental stiffness matrices are directly assembled into the original global stiffness matrix.

Taking the frame structure as shown in figure 8.4 again as an example, the elemental locating vectors for the two elements are

$$\xi^{\textcircled{1}} = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6]^T, \quad \xi^{\textcircled{2}} = [1 \quad 2 \quad 3 \quad 7 \quad 8 \quad 9]^T$$

Then, the stiffness matrices of elements ① and ② are assembled into the global stiffness matrix by using the direct stiffness method. For element ①, the coefficient $k_{ij}^{\textcircled{1}}$ of elemental stiffness matrix occupies the $\xi^{\textcircled{1}}(i)$ th row and $\xi^{\textcircled{1}}(j)$ th column in the global stiffness matrix. As a result, in terms of the following numbering way for arranging the positions of matrix coefficients the contribution matrix of element ① is given by

$$K^{\textcircled{1}} = \begin{matrix} & \begin{matrix} (1) & (2) & (3) & & (4) & (5) & (6) \\ & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ & 1 & 2 & 3 & & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} (1) \rightarrow 1 \\ (2) \rightarrow 2 \\ (3) \rightarrow 3 \end{matrix} & \left[\begin{array}{ccc|ccc|ccc} 300 & 0 & 0 & -300 & 0 & 0 & \otimes & \otimes & \otimes \\ 0 & 12 & 30 & 0 & -12 & 30 & \otimes & \otimes & \otimes \\ 0 & 30 & 100 & 0 & -30 & 50 & \otimes & \otimes & \otimes \end{array} \right] \\ & \hline \begin{matrix} (4) \rightarrow 4 \\ (5) \rightarrow 5 \\ (6) \rightarrow 6 \end{matrix} & \left[\begin{array}{ccc|ccc|ccc} -300 & 0 & 0 & 300 & 0 & 0 & \otimes & \otimes & \otimes \\ 0 & -12 & -30 & 0 & 12 & -30 & \otimes & \otimes & \otimes \\ 0 & 30 & 50 & 0 & -30 & 100 & \otimes & \otimes & \otimes \end{array} \right] \\ & \hline & \left[\begin{array}{ccc|ccc|ccc} \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \end{array} \right] \end{matrix} \times 10^4$$

where $\otimes$ represents that the contribution of the position of elemental stiffness matrix in global stiffness matrix is zero.

For element ②, the row and column numbers of the coefficient of elemental stiffness matrix in global stiffness matrix are $\zeta^{(2)}(i)$ and $\zeta^{(2)}(j)$, respectively. Similarly, the contribution matrix from element ② is written as

$$\mathbf{K}^{(2)} = \begin{array}{c} \begin{array}{ccc} (1) & (2) & (3) \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \\ \begin{array}{ccc} (1) \rightarrow 1 \\ (2) \rightarrow 2 \\ (3) \rightarrow 3 \end{array} \end{array} \left[\begin{array}{ccc|ccc|ccc} 12 & 0 & -30 & \otimes & \otimes & \otimes & -12 & 0 & -30 \\ 0 & 300 & 0 & \otimes & \otimes & \otimes & 0 & -300 & 0 \\ -30 & 0 & 100 & \otimes & \otimes & \otimes & 30 & 0 & 50 \\ \hline \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \hline -12 & 0 & 30 & \otimes & \otimes & \otimes & 12 & 0 & 30 \\ 0 & -300 & 0 & \otimes & \otimes & \otimes & 0 & 300 & 0 \\ -30 & 0 & 50 & \otimes & \otimes & \otimes & 30 & 0 & 100 \end{array} \right] \times 10^4$$

The global stiffness matrix of the frame structure is the sum of the contribution matrices of two elements, namely

$$\mathbf{K} = \mathbf{K}^{(1)} + \mathbf{K}^{(2)}$$

$$= 10^4 \times \left[\begin{array}{ccc|ccc|ccc} 312 & 0 & -30 & -300 & 0 & 0 & -12 & 0 & -30 \\ 0 & 312 & 30 & 0 & -12 & 30 & 0 & -300 & 0 \\ -30 & 30 & 200 & 0 & -30 & 50 & 30 & 0 & 50 \\ \hline -300 & 0 & 0 & 300 & 0 & 0 & \otimes & \otimes & \otimes \\ 0 & -12 & -30 & 0 & 12 & -30 & \otimes & \otimes & \otimes \\ 0 & 30 & 50 & 0 & -30 & 100 & \otimes & \otimes & \otimes \\ \hline -12 & 0 & 30 & \otimes & \otimes & \otimes & 12 & 0 & 30 \\ 0 & -300 & 0 & \otimes & \otimes & \otimes & 0 & 300 & 0 \\ -30 & 0 & 50 & \otimes & \otimes & \otimes & 30 & 0 & 100 \end{array} \right]$$

Now, there are still some marks $\otimes$ in the global stiffness matrix, which means that there is no direct relationship between some of the nodes in the structure. From the frame in figure 8.4, it is seen that nodes 2 and 3 are not connected by a member, hence the corresponding positions in the global stiffness matrix can be neglected and should be set to 0. In this way, the original global stiffness matrix of the structure is determined by

$$\mathbf{K} = 10^4 \times \left[\begin{array}{ccc|ccc|ccc}
 312 & 0 & -30 & -300 & 0 & 0 & -12 & 0 & -30 \\
 0 & 312 & 30 & 0 & -12 & 30 & 0 & -300 & 0 \\
 -30 & 30 & 200 & 0 & -30 & 50 & 30 & 0 & 50 \\
 \hline
 -300 & 0 & 0 & 300 & 0 & 0 & 0 & 0 & 0 \\
 0 & -12 & -30 & 0 & 12 & -30 & 0 & 0 & 0 \\
 0 & 30 & 50 & 0 & -30 & 100 & 0 & 0 & 0 \\
 \hline
 -12 & 0 & 30 & 0 & 0 & 0 & 12 & 0 & 30 \\
 0 & -300 & 0 & 0 & 0 & 0 & 0 & 300 & 0 \\
 -30 & 0 & 50 & 0 & 0 & 0 & 30 & 0 & 100
 \end{array} \right] \quad (8.37)$$

8.4.3 Imposing Support Conditions

Since node 2 is a pin support, all the displacement components except nodal rotation (global numberings 4 and 5) on it should be zero. Node 3 is fixed support, so all the displacement components (global numberings 7, 8, and 9) should be zero. For the case with known displacement components being zero, we can delete the corresponding rows and columns from the global stiffness matrix to achieve a reduced global stiffness matrix:

$$\mathbf{K} = 10^4 \times \left[\begin{array}{cccc}
 312 & 0 & -30 & 0 \\
 0 & 312 & 30 & 30 \\
 -30 & 30 & 200 & 50 \\
 0 & 30 & 50 & 100
 \end{array} \right] \quad (8.38)$$

Such a method handling the constraints after forming the global stiffness matrix is termed as the **post-treatment method**. Since the size of the original matrix is reduced, the original numberings of rows and columns in the global stiffness matrix are changed as well. Consequently, other methods such as the setting large number method and the multiplying large number method are normally used in practice to introduce the constraint conditions of the structure. The global stiffness matrix treated by these methods is not reduced, and the size and numberings remain unchanged, which belongs to the post-treatment method.

In fact, the constraint conditions of the supports can also be treated in the early stages. A feasible way is to set the global numberings of nodal displacement components corresponding to zero displacement constraints to be 0. All the members in the rows and columns of the elemental stiffness matrix corresponding to 0 in the elemental locating vector are not assembled into the global stiffness matrix. The benefit of doing this is that the size of the global stiffness matrix can be reduced, and the required computer memory space is also decreased. Such a method for treating constraint conditions is termed the **pre-treatment method**.

Herein, the theoretical basis of the pretreatment method is elaborated. Considering the constraint boundary condition of the supports, the nodal displacement vector can be divided into two groups, *i.e.*, Δ_a (unknown displacement vector) and Δ_b (zero displacement vector, $\Delta_b = \mathbf{0}$). Similarly, the nodal force vector of structure can also be divided into two groups, *i.e.*, F_a (known nodal force vector) and F_b (unknown nodal force vector). They satisfy the following stiffness equation

$$\begin{bmatrix} F_a \\ F_b \end{bmatrix} = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix}$$

The above stiffness matrix is partitioned into the sub-matrices corresponding to unknown and known displacement vectors. Considering that $\Delta_b = 0$, we have

$$F_a = K_{aa}\Delta_a$$

Clearly, if the constraint conditions at the supports are treated in advance, the original stiffness matrix is reduced, and the reduced stiffness matrix is composed of the stiffness coefficients corresponding to the unknown nodal displacements.

No matter which method is used, the pretreatment method or post-treatment method, the obtained global stiffness matrices are identical to each other. The post-treatment method is easier to understand and thus preferred for teaching. However, the pretreatment method is more widely adopted in commercial computer programs for large-scale structural analysis.

8.4.4 Properties of Global Stiffness Matrix

(1) Meaning of global stiffness coefficient

The element K_{ij} in a global stiffness matrix K is termed the global stiffness coefficient. It represents the i th nodal force F_i when the j th nodal displacement component is 1 (all the other nodal displacement components are 0).

(2) Global stiffness matrix is symmetric, *i.e.*,

$$K_{ij} = K_{ji} \quad (8.39)$$

(3) Global stiffness matrix is invertible and positive definite when sufficient displacement constraint conditions of supports are imposed.

(4) With proper node numbering, the global stiffness matrix is a banded and sparse matrix. Although the illustrative example in this section involves the analysis of frame structure with only 2 elements, for the computation of complex frames and continuous beams with many elements the global stiffness matrix is banded and sparse.

8.4.5 Treatment of Pinned Joints

The frame structure with pinned joints shown in figure 8.7 is used as an example to discuss some related methods for handling pinned joints. The node and element numberings of the structure are identical to the frame without any pinned joint. Based on the above-mentioned ways, the frame structure can be divided into 3 elements, *i.e.*, ①, ②, and ③ and 4 nodes as displayed in figure 8.7. According to the pretreatment method, the nodal displacement components at fixed supports C and D are set to (0, 0, 0). For each element, the local numbering remains unchanged, which starts from (1) to (6) in sequence. Nevertheless, global numbering should be paid attention. Since the 2 angular displacements of member ends connected with the pinned joint B are different, thus it can be regarded as two half-independent nodes (B_1 and B_2). Their linear displacements are the same which is not independent, but their angular displacements are different which is independent. Hence, their linear displacements should adopt the same global numbering, while their angular displacements should use different global numberings. For example, the global numbering for node B_1 is (4, 5, 6) and for B_2 it is (4, 5, 7). In the process of assembling the elemental stiffness matrix, it is necessary to consider the corresponding global numbering of each element and elemental locating vector and take the elements into the global stiffness matrix in terms of the respective numberings and positions.

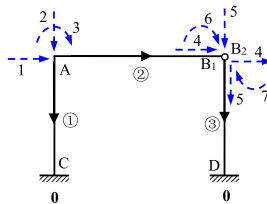


FIG. 8.7 – Frame structure with pinned joints.

8.5 Equivalent Nodal Loads

8.5.1 Basic Equation of Matrix Displacement Method

In section 8.4, the global stiffness matrix $\mathbf{K}$ is discussed and the global stiffness equation is established

$$\mathbf{F} = \mathbf{K}\mathbf{A} \quad (8.40)$$

Note that the global stiffness equation (8.40) is constructed based on the primary system of the displacement method of the original structure, This equation represents the formula to calculate nodal forces $\mathbf{F}$ (*i.e.*, the restraint forces resulting from

additional constraints in primary system) with nodal displacements $\mathbf{A}$, and only reflects the stiffness properties of a structure while does not involve real loads acting on the structure. Equation (8.40) is not the basic equation of the displacement method for analyzing the original structure.

In order to construct the basic equation of the matrix displacement method, let us recall the derivation approach in section 6.6 and consider the two states of the primary system of displacement method separately.

- (1) Assuming the loads are solely acting on the structure (nodal displacement vector $\mathbf{A}$ is set as 0), the resultant nodal force vector of primary structure is denoted by $\mathbf{F}_P$.
- (2) Assuming the nodal displacements $\mathbf{A}$ are solely acting on the structure (the loads are taken as 0), the resultant nodal force vector is denoted by $\mathbf{F} = \mathbf{K}\mathbf{A}$.

The basic equation of the matrix displacement method should be formulated as

$$\mathbf{F} + \mathbf{F}_P = \mathbf{0}$$

namely,

$$\mathbf{K}\mathbf{A} + \mathbf{F}_P = \mathbf{0} \quad (8.41)$$

8.5.2 Equivalent Nodal Loads of Elements

In structural analysis using the matrix displacement method, in addition to assembling the global stiffness matrix and imposing the supporting constraints the loads acting on the structure must also be considered. The bearing external force of the nodes in the structure is called the nodal load, and the bearing external force of a member between two nodes is termed the inter-node load. In the matrix displacement method, the forces and displacements are all discussed on the nodes, hence the inter-node loads need to be transformed into equivalent nodal loads. The rule for such an equivalent transformation is that the resultant nodal constraint forces on the primary system are identical for the two kinds of loads.

If the nodal constraint force vector in the primary structure resulting from the original inter-node load is $\mathbf{F}_P$, then the **equivalent nodal loads** $\mathbf{P}$ are the negative of $\mathbf{F}_P$

$$\mathbf{P} = -\mathbf{F}_P \quad (8.42)$$

Substituting equation (8.42) into equation (8.41), the basic equation of matrix displacement method is written as

$$\mathbf{K}\mathbf{A} = \mathbf{P} \quad (8.43)$$

It is seen from equations (8.40) and (8.43) that, if the nodal constraint forces $\mathbf{F}$ in global stiffness equation (8.40) is replaced by the equivalent nodal loads $\mathbf{P}$, the basic equation (8.43) of matrix displacement method, namely, the structural stiffness equation, is obtained immediately.

In what follows, the fixed-end constraint forces (nodal constraint forces) of member elements under some typical inter-node loads are given, which are taken minus and become the equivalent nodal loads of the element.

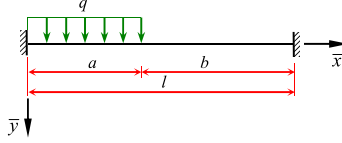


FIG. 8.8 – Distributed load at a member element.

For the distributed load shown in figure 8.8, the fixed-end constraint forces of member element are written by

Left node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= -qa \left(1 - \frac{a^2}{l^2} + \frac{a^3}{2l^3} \right) \\ \bar{M}_P &= -\frac{qa^2}{12} \left(6 - 8\frac{a}{l} + 3\frac{a^2}{l^2} \right)\end{aligned}$$

Right node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= -q \frac{a^3}{l^2} \left(1 - \frac{a}{2l} \right) \\ \bar{M}_P &= \frac{qa^3}{12l} \left(4 - 3\frac{a}{l} \right)\end{aligned}$$

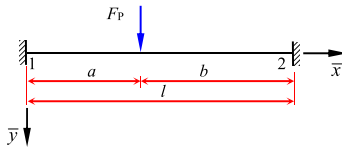


FIG. 8.9 – Concentrated load at a member element.

For the concentrated load shown in figure 8.9, the fixed-end constraint forces of member element are expressed as follows.

Left node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= -F_P \frac{b^2}{l^2} \left(1 + 2\frac{a}{l} \right) \\ \bar{M}_P &= -F_P \frac{ab^2}{l^2}\end{aligned}$$

Right node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= -F_P \frac{a^2}{l^2} \left(1 + 2\frac{b}{l}\right) \\ \bar{M}_P &= F_P \frac{ab^2}{l^2}\end{aligned}$$

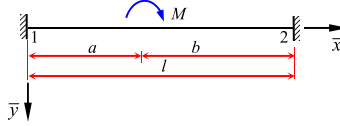


FIG. 8.10 – Concentrated couple moment at a member element.

For the concentrated couple moment in figure 8.10, the fixed-end constraint forces of member element are:

Left node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= 6 \frac{Mab}{l^3} \\ \bar{M}_P &= M \frac{b}{l} \left(2 - 3\frac{b}{l}\right)\end{aligned}$$

Right node:

$$\begin{aligned}\bar{F}_{xP} &= 0 \\ \bar{F}_{yP} &= -6 \frac{Mab}{l^3} \\ \bar{M}_P &= M \frac{a}{l} \left(2 - 3\frac{a}{l}\right)\end{aligned}$$

8.5.3 Equivalent Nodal Loads of Structure

In subsection 8.5.2, the equivalent nodal forces of member elements under the inter-node loads are calculated, which is actually examined in the local coordinate system. When performing structural analysis, they should be transformed into the global coordinate system and assembled into the global load vector just like the assembling of the elemental stiffness matrix. Firstly, the equivalent nodal load vector of an element in the local coordinate system is calculated: $\bar{\mathbf{P}}^e = -\bar{\mathbf{F}}_P^e$. It is transformed into the global coordinate system, and the corresponding equivalent nodal load vector $\mathbf{P}^e$ is given by

$$\mathbf{P}^e = \mathbf{T}^T \bar{\mathbf{P}}^e = -\mathbf{T}^T \bar{\mathbf{F}}_P^e \quad (8.44)$$

Then, all the elemental nodal loads are assembled into the global nodal load vector in terms of respective numbering based on the elemental locating vector.

Finally, the member end force of each element is the sum of the fixed end force and the resultant member end forces from equivalent nodal loads

$$\bar{\mathbf{F}}^e = \bar{\mathbf{F}}_P^e + \bar{\mathbf{k}}^e \bar{\mathbf{A}}^e \quad (8.45)$$

Example 8.2. Calculate the equivalent nodal load vector $\mathbf{P}$ of the frame structure for example 8.1 as shown in figure 8.4 under the given loads displayed in figure 8.11.

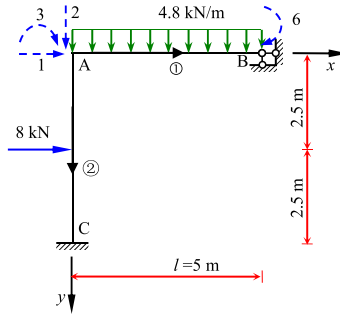


FIG. 8.11 – A frame structure under given loads.

Solution: (1) Calculate the fixed-end constraint force vector $\bar{\mathbf{F}}_P^e$ of two elements in local coordinate system.

For element ①, in terms of relevant formulation the fixed-end constraint force vector in local coordinate system is

$$\bar{\mathbf{F}}_P^{①} = \begin{bmatrix} \bar{F}_{xP1} \\ \bar{F}_{yP1} \\ \bar{M}_{P1} \\ \bar{F}_{xP2} \\ \bar{F}_{yP2} \\ \bar{M}_{P2} \end{bmatrix} = \begin{bmatrix} 0 \\ -12 \text{ kN} \\ -10 \text{ kN}\cdot\text{m} \\ 0 \\ -12 \text{ kN} \\ 10 \text{ kN}\cdot\text{m} \end{bmatrix}$$

For element ②

$$\bar{\mathbf{F}}_P^{②} = \begin{bmatrix} \bar{F}_{xP1} \\ \bar{F}_{yP1} \\ \bar{M}_{P1} \\ \bar{F}_{xP2} \\ \bar{F}_{yP2} \\ \bar{M}_{P2} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \text{ kN} \\ 5 \text{ kN}\cdot\text{m} \\ 0 \\ 4 \text{ kN} \\ -5 \text{ kN}\cdot\text{m} \end{bmatrix}$$

- (2) Solve the equivalent nodal load vector $\mathbf{P}^e$ of two elements in global coordinate system.

For element ①, in terms of equation (8.44) we have

$$\mathbf{P}^{\textcircled{1}} = \begin{bmatrix} F_{xP1} \\ F_{yP1} \\ M_{P1} \\ F_{xP2} \\ F_{yP2} \\ M_{P2} \end{bmatrix} = -\mathbf{T}^{\textcircled{1}T} \begin{bmatrix} \bar{F}_{xP1} \\ \bar{F}_{yP1} \\ \bar{M}_{P1} \\ \bar{F}_{xP2} \\ \bar{F}_{yP2} \\ \bar{M}_{P2} \end{bmatrix} = -\mathbf{I} \begin{bmatrix} 0 \\ 12 \\ 10 \\ 0 \\ 12 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 10 \\ 0 \\ 12 \\ -10 \end{bmatrix}$$

For element ②

$$\mathbf{P}^{\textcircled{2}} = \begin{bmatrix} F_{xP1} \\ F_{yP1} \\ M_{P1} \\ - \\ F_{xP2} \\ F_{yP2} \\ M_{P2} \end{bmatrix} = -\mathbf{T}^{\textcircled{2}T} \begin{bmatrix} \bar{F}_{xP1} \\ \bar{F}_{yP1} \\ \bar{M}_{P1} \\ - \\ \bar{F}_{xP2} \\ \bar{F}_{yP2} \\ \bar{M}_{P2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & | & 0 & -1 & 0 \\ 0 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ -5 \\ - \\ 0 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -5 \\ - \\ 4 \\ 0 \\ 5 \end{bmatrix}$$

- (3) Compute the equivalent nodal load vector $\mathbf{P}$ of the frame.

Assemble the nodal load vectors of elements into the global nodal load vector. Similar to the process of assembling the elemental stiffness matrix into the global stiffness matrix, the positions of nodal load vectors should also be determined *via* the elemental locating vector, and then added one by one in the global nodal load vector.

For element ①, the locating vector is $\xi^{\textcircled{1}} = [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T$, and the contribution of nodal load vector of element in the global nodal load vector is

$$\mathbf{P}_1 = [0 \ 12 \ 10 \ 0 \ 12 \ -10 \ \otimes \ \otimes \ \otimes]^T$$

For element ②, the locating vector is $\xi^{\textcircled{2}} = [1 \ 2 \ 3 \ 7 \ 8 \ 9]^T$ and the contribution of nodal load vector of element in the global nodal load vector is given by

$$\mathbf{P}_2 = [4 \ 0 \ -5 \ \otimes \ \otimes \ \otimes \ 4 \ 0 \ 5]^T$$

Adding the two vectors of contribution leads to the global nodal load vector

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 = [4 \ 12 \ 5 \ 0 \ 12 \ -10 \ 4 \ 0 \ 5]^T$$

Since for nodes 2 and 3, there exists the case that the known displacement components are zero (of which the global numberings are 4, 5, 7, 8, and 9), the corresponding elements in the global stiffness matrix have been removed, and similarly, the respective elements in the global nodal load vector also need to be removed. Finally, the global nodal load vector of the frame structure is achieved

$$\mathbf{P} = \begin{bmatrix} 4 \text{ kN} \\ 12 \text{ kN} \\ 5 \text{ kN}\cdot\text{m} \\ -10 \text{ kN}\cdot\text{m} \end{bmatrix}$$

8.6 Computational Procedures and Examples

In the matrix displacement method, after obtaining the global stiffness matrix and global nodal load vector only nodal displacements are unknown, and the nodal displacements and nodal loads satisfy the basic equation (8.43). For such a set of linear algebraic equations, they can be solved by means of matrix algebraic operation to give all the elements in the nodal displacement vector $\mathbf{A}$:

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{P}$$

When implementing structural analysis by the matrix displacement method, a unified and standard procedure can be followed: assemble the elemental stiffness matrix and nodal load vector into the global ones, and then solve the linear algebraic equations to obtain the nodal displacements. Thus, the matrix displacement method is very suitable to be implemented with computer programming.

For the example illustrated in figure 8.11, the nodal displacements can be directly computed as

$$\begin{aligned} \mathbf{A} = \mathbf{K}^{-1}\mathbf{P} &= 10^{-4} \times \begin{bmatrix} 32.59 & -0.28 & 5.61 & -2.72 \\ -0.28 & 33.15 & -2.89 & -8.50 \\ 5.61 & -2.89 & 58.35 & -28.31 \\ -2.72 & -8.50 & -28.31 & 116.70 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 5 \\ -10 \end{bmatrix} \\ &= 10^{-2} \times \begin{bmatrix} 1.82 \\ 4.67 \\ 5.63 \\ -14.21 \end{bmatrix} \end{aligned}$$

Considering the case that the known nodal displacement components are 0, all the displacement components in the global coordinate system read

$$\mathbf{A} = \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix} = 10^{-2} \times \begin{bmatrix} 1.82 \\ 4.67 \\ 5.63 \\ 0 \\ 0 \\ -14.21 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In summary, the computational procedures of the matrix displacement method are presented as follows:

- (1) Collect the original data, assign the element and node numbers for member structure.
- (2) Form the elemental stiffness matrix $\bar{\mathbf{k}}^e$ in local coordinate system.
- (3) Form the elemental stiffness matrix $\mathbf{k}^e$ in global coordinate system.
- (4) Assemble all the elemental stiffness matrices to form the global stiffness matrix with the direct stiffness method.
- (5) Solve the equivalent nodal load vectors of member elements under the local coordinate system, transform them into the equivalent nodal load vectors of elements $\mathbf{P}^e$ in global coordinate system, and then assemble them to form the global nodal load vector $\mathbf{P}$.
- (6) Solve the equation $\mathbf{KA} = \mathbf{P}$ to get the nodal displacement vector $\mathbf{A}$.
- (7) Take the obtained nodal displacements into each element to calculate the member end forces.

In what follows, the matrix displacement method is applied for internal force analysis of some common structures.

8.6.1 Example of Truss Analysis

Example 8.3. Solve the internal forces of each element of the truss structure shown in figure 8.12. EA for all the members is identical.

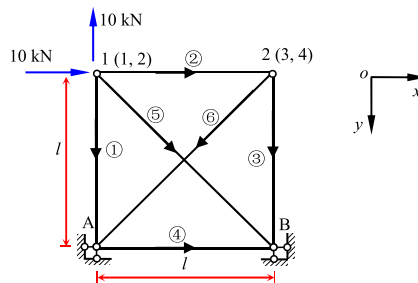


FIG. 8.12 – A truss structure subjected to specified loads.

Solution: (1) Collect the original data and assign the element and node numbers.

The geometric information as well as the node and element numberings are shown in figure 8.12, and the elastic modulus of the material is denoted by E . The global numberings of displacement components of node 1 are (1, 2) and the counterparts of node 2 are (3, 4). According to the pretreatment method, the corresponding numberings of nodal displacement components of two pin supports are 0.

(2) Form the elemental stiffness matrices in a local coordinate system.

Firstly, according to equation (8.30) the elemental stiffness matrix of truss bars in local coordinate system can be calculated. Because the lengths of elements ①, ②, ③ and ④ are all equal to l , the stiffness matrices of these elements under the local coordinate system are given by

$$\bar{\mathbf{k}}^{①} = \bar{\mathbf{k}}^{②} = \bar{\mathbf{k}}^{③} = \bar{\mathbf{k}}^{④} = \frac{EA}{l} \times \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

For elements ⑤ and ⑥, the lengths of them are $\sqrt{2}l$. So, we have

$$\bar{\mathbf{k}}^{⑤} = \bar{\mathbf{k}}^{⑥} = \frac{EA}{\sqrt{2}l} \times \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(3) Calculate the elemental stiffness matrices in global coordinate system.

Then, in terms of equation (8.33) elemental stiffness matrices in global coordinate system are obtained *via* coordinate transformation. For elements ② and ④, the angle between the two coordinate systems is $\alpha = 0$, so the coordinate transformation matrix is an identity matrix

$$\mathbf{T} = \left[\begin{array}{cc|cc} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 1 & 0 \\ & & 0 & 1 \end{array} \right]$$

And thus the elemental stiffness matrix in global coordinate system is

$$\mathbf{k}^{②} = \mathbf{k}^{④} = \mathbf{T}^T \bar{\mathbf{k}}^{②} \mathbf{T} = \frac{EA}{l} \times \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

For elements ① and ③, the angle is $\alpha = \pi/2$, and the coordinate transformation matrix is written as

$$\mathbf{T} = \left[\begin{array}{cc|cc} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 0 & 1 \\ & & -1 & 0 \end{array} \right]$$

So, we get

$$\mathbf{k}^{\textcircled{1}} = \mathbf{k}^{\textcircled{3}} = \mathbf{T}^T \bar{\mathbf{k}}^{\textcircled{1}} \mathbf{T} = \frac{EA}{l} \times \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right]$$

For element ⑤ of which the angle is $\alpha = \pi/4$, the coordinate transformation matrix is

$$\mathbf{T} = \frac{1}{\sqrt{2}} \times \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array} \right]$$

We have

$$\mathbf{k}^{\textcircled{5}} = \mathbf{T}^T \bar{\mathbf{k}}^{\textcircled{5}} \mathbf{T} = \frac{EA}{l} \times \frac{1}{2\sqrt{2}} \times \left[\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array} \right]$$

For element ⑥ of which the angle is $\alpha = 3\pi/4$, the coordinate transformation matrix is given by

$$\mathbf{T} = \frac{1}{\sqrt{2}} \times \left[\begin{array}{cc|cc} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{array} \right]$$

And we get

$$\mathbf{k}^{\textcircled{6}} = \mathbf{T}^T \bar{\mathbf{k}}^{\textcircled{6}} \mathbf{T} = \frac{EA}{l} \times \frac{1}{2\sqrt{2}} \times \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ \hline -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right]$$

(4) Form the global elemental stiffness matrix with direct stiffness method.

In terms of the respective element locating vectors, the elemental stiffness matrices are added to the global stiffness matrix one by one. Because the two supports of the truss in this example are pin supports, the two linear displacements are all equal to 0, and the corresponding rows and columns in elemental stiffness matrices are not assembled into the global stiffness matrix.

The element locating vectors for the elements in figure 8.12 are expressed separately

$$\xi^{①} = [1 \quad 2 \quad 0 \quad 0]^T \quad \xi^{②} = [1 \quad 2 \quad 3 \quad 4]^T$$

$$\xi^{③} = [3 \quad 4 \quad 0 \quad 0]^T \quad \xi^{④} = [0 \quad 0 \quad 0 \quad 0]^T$$

$$\xi^{⑤} = [1 \quad 2 \quad 0 \quad 0]^T \quad \xi^{⑥} = [3 \quad 4 \quad 0 \quad 0]^T$$

And the global stiffness matrix of truss is obtained

$$\mathbf{K} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \left[\begin{array}{cc|cc} 1.35 & 0.35 & -1 & 0 \\ 0.35 & 1.35 & 0 & 0 \\ \hline -1 & 0 & 1.35 & -0.35 \\ 0 & 0 & -0.35 & 1.35 \end{array} \right] \frac{EA}{l} \end{matrix}$$

(5) Compute nodal load vector $\mathbf{P}$.

There are two external forces acting on node 1, and the nodal load vector of the structure is

$$\mathbf{P} = \begin{bmatrix} 10 \\ -10 \\ -- \\ 0 \\ 0 \end{bmatrix}$$

(6) Solve the basic equation.

$$\frac{EA}{l} \left[\begin{array}{cc|cc} 1.35 & 0.35 & -1 & 0 \\ 0.35 & 1.35 & 0 & 0 \\ \hline -1 & 0 & 1.35 & -0.35 \\ 0 & 0 & -0.35 & 1.35 \end{array} \right] \begin{bmatrix} u_1 \\ v_1 \\ -- \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ -- \\ 0 \\ 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} u_1 \\ v_1 \\ -- \\ u_2 \\ v_2 \end{bmatrix} = \frac{l}{EA} \begin{bmatrix} 26.94 \\ -14.42 \\ -- \\ 21.36 \\ 5.58 \end{bmatrix}$$

- (7) Solve the nodal forces under the local coordinate systems, *i.e.*, member end force vectors.

For element ①

$$\begin{aligned}\bar{\mathbf{F}}^{\textcircled{1}} &= \mathbf{TF}^{\textcircled{1}} = \mathbf{Tk}^{\textcircled{1}}\Delta^{\textcircled{1}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 26.94 \\ -14.42 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -14.4 \\ 0 \\ 14.4 \\ 0 \end{bmatrix}\end{aligned}$$

For element ②

$$\bar{\mathbf{F}}^{\textcircled{2}} = \mathbf{TF}^{\textcircled{2}} = \mathbf{Tk}^{\textcircled{2}}\Delta^{\textcircled{2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 26.94 \\ -14.42 \\ 21.36 \\ 5.58 \end{bmatrix} = \begin{bmatrix} 5.58 \\ 0 \\ -5.58 \\ 0 \end{bmatrix}$$

For element ③

$$\begin{aligned}\bar{\mathbf{F}}^{\textcircled{3}} &= \mathbf{TF}^{\textcircled{3}} = \mathbf{Tk}^{\textcircled{3}}\Delta^{\textcircled{3}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 21.36 \\ 5.58 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5.58 \\ 0 \\ -5.58 \\ 0 \end{bmatrix}\end{aligned}$$

For element ④, the axial displacement is zero, so the axial force is also zero.

For element ⑤

$$\begin{aligned}\bar{\mathbf{F}}^{\textcircled{5}} &= \mathbf{TF}^{\textcircled{5}} = \mathbf{Tk}^{\textcircled{5}}\Delta^{\textcircled{5}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ &\quad \times \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 26.94 \\ -14.42 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.26 \\ 0 \\ -6.26 \\ 0 \end{bmatrix}\end{aligned}$$

For element ⑥

$$\begin{aligned}\bar{\mathbf{F}}^{\textcircled{6}} &= \mathbf{TF}^{\textcircled{6}} = \mathbf{T}\mathbf{k}^{\textcircled{6}}\mathbf{A}^{\textcircled{6}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \\ &\times \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 21.36 \\ 5.58 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -7.89 \\ 0 \\ 7.89 \\ 0 \end{bmatrix}\end{aligned}$$

8.6.2 Example of Frame Structure

Example 8.4. Solve the nodal displacements of frame structure exhibited in figure 8.13. The cross-sections of all the members are rectangle. The cross-section area is $b_2 \cdot h_2 = 0.5 \times 1.26 \text{ m}^2 = 0.63 \text{ m}^2$ for the beam and $b_1 \cdot h_1 = 0.5 \times 1 \text{ m}^2 = 0.5 \text{ m}^2$ for the columns. Elastic modulus is $E = 1$.

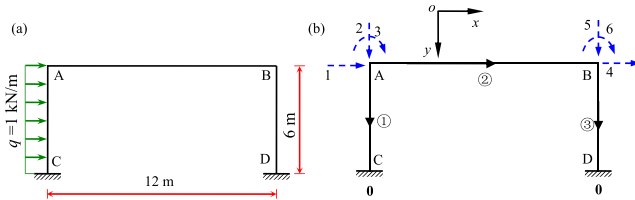


FIG. 8.13 – (a) A frame structure under uniformly distributed load, (b) element and node numberings.

Solution: (1) Conduct the element and node numberings.

The numberings of the element and nodes are shown in figure 8.13. The global numberings of displacement components of node A are (1, 2, 3), and those numberings of node B are (3, 4, 5). The numberings of nodal displacement components of the fixed supports C and D are all 0. Related geometrical and physical parameters of the frame are calculated as follows.

For the column

$$A_1 = 0.5 \text{ m}^2, I_1 = \frac{1}{24} \text{ m}^4$$

$$l_1 = 6 \text{ m}, \frac{EI_1}{l_1} = 6.94 \times 10^{-3}$$

$$\frac{EA_1}{l_1} = 83.3 \times 10^{-3}, \frac{2EI_1}{l_1} = 13.9 \times 10^{-3}$$

$$\frac{4EI_1}{l_1} = 27.8 \times 10^{-3}, \frac{6EI_1}{l_1^2} = 6.94 \times 10^{-3}$$

$$\frac{12EI_1}{l_1^3} = 2.31 \times 10^{-3}$$

For the beam

$$A_2 = 0.63 \text{ m}^2, I_2 = \frac{1}{12} \text{ m}^4, l^2 = 12 \text{ m}$$

$$\frac{EA_2}{l_2} = 52.5 \times 10^{-3}, \frac{EI_2}{l_2} = 6.94 \times 10^{-3}$$

$$\frac{2EI_2}{l_2} = 13.9 \times 10^{-3}, \frac{4EI_2}{l_2} = 27.8 \times 10^{-3}$$

$$\frac{6EI_2}{l_2^2} = 3.47 \times 10^{-3}, \frac{12EI_2}{l_2^3} = 0.58 \times 10^{-3}$$

(2) Form the elemental stiffness matrices.

Calculate the elemental stiffness matrices under the local coordinate system. For elements ① and ③:

$$\bar{\mathbf{k}}^{①} = 10^{-3} \times \begin{bmatrix} 83.3 & 0 & 0 & -83.3 & 0 & 0 \\ 0 & 2.31 & 6.94 & 0 & -2.31 & 6.94 \\ 0 & 6.94 & 27.8 & 0 & -6.94 & 13.9 \\ -83.3 & 0 & 0 & 83.3 & 0 & 0 \\ 0 & -2.31 & -6.94 & 0 & 2.31 & -6.94 \\ 0 & 6.94 & 13.9 & 0 & -6.94 & 27.8 \end{bmatrix}$$

For element ②:

$$\bar{\mathbf{k}}^{②} = 10^{-3} \times \begin{bmatrix} 52.5 & 0 & 0 & -52.5 & 0 & 0 \\ 0 & 0.58 & 3.47 & 0 & -0.58 & 3.47 \\ 0 & 3.47 & 27.8 & 0 & -3.47 & 13.9 \\ -52.5 & 0 & 0 & 52.5 & 0 & 0 \\ 0 & -0.58 & -3.47 & 0 & 0.58 & -3.47 \\ 0 & 3.47 & 13.9 & 0 & -3.47 & 27.8 \end{bmatrix}$$

Calculate the elemental stiffness matrices under the global coordinate system. For elements ① and ③, the coordinate transformation matrix ($\alpha = \pi/2$) is

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array} \right] \quad \mathbf{0}$$

Therefore,

$$\mathbf{k}^{\textcircled{1}} = \mathbf{k}^{\textcircled{3}} = \mathbf{T}^T \mathbf{k}^{\textcircled{1}} \mathbf{T} = 10^{-3} \times \left[\begin{array}{cccccc} 2.31 & 0 & -6.94 & -2.31 & 0 & -6.94 \\ 0 & 83.3 & 0 & 0 & -83.3 & 0 \\ -6.94 & 0 & 27.8 & 6.94 & 0 & 13.9 \\ -2.31 & 0 & 6.94 & 2.31 & 0 & 6.94 \\ 0 & -83.3 & 0 & 0 & 83.3 & 0 \\ -6.94 & 0 & 13.9 & 6.94 & 0 & 27.8 \end{array} \right]$$

For element ②, the angle $\alpha = 0$ makes $\mathbf{T} = \mathbf{I}$. The elemental stiffness matrices under global and local coordinate systems are identical for this case.

$$\mathbf{k}^{\textcircled{2}} = \bar{\mathbf{k}}^{\textcircled{2}}$$

(3) Assemble the global stiffness matrix $\mathbf{K}$ with the direct stiffness method.

In terms of the respective element locating vectors, the elemental stiffness matrices are assembled into global stiffness matrix one by one. According to the pretreatment method, the rows and columns in the global stiffness matrix corresponding to zero displacement components are not considered.

The element locating vectors of the members in figure 8.13 are written separately by

$$\boldsymbol{\xi}^{\textcircled{1}} = [1 \quad 2 \quad 3 \quad 0 \quad 0 \quad 0]^T$$

$$\boldsymbol{\xi}^{\textcircled{2}} = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6]^T$$

$$\boldsymbol{\xi}^{\textcircled{3}} = [4 \quad 5 \quad 6 \quad 0 \quad 0 \quad 0]^T$$

And the global stiffness matrix of frame is obtained, namely

$$\mathbf{K} = \left[\begin{array}{ccc|ccc} 54.81 & 0 & -6.94 & -52.5 & 0 & 0 \\ 0 & 83.88 & 3.47 & 0 & -0.58 & 3.47 \\ -6.94 & 3.47 & 55.6 & 0 & -3.74 & 13.9 \\ \hline -52.5 & 0 & 0 & 54.81 & 0 & -6.94 \\ 0 & -0.58 & -3.47 & 0 & 83.88 & -3.47 \\ 0 & 3.47 & 13.9 & -6.94 & -3.47 & 55.6 \end{array} \right] \times 10^{-3}$$

(4) Compute equivalent nodal load vector $\mathbf{P}$.

For inter-node load, only element ① is subjected to a uniformly distributed load, and the fixed-end constraint force vector under the local coordinate system is given by

$$\bar{\mathbf{F}}_P^{\textcircled{1}} = \begin{bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{M}_1 \\ \bar{F}_{x2} \\ \bar{F}_{y2} \\ \bar{M}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

It is transformed into the global coordinate system which reads

$$\mathbf{P}^{\textcircled{1}} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ F_{y2} \\ M_2 \end{bmatrix} = -\mathbf{T}^{\textcircled{1}\text{T}} \begin{bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{M}_1 \\ \bar{F}_{x2} \\ \bar{F}_{y2} \\ \bar{M}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ -3 \\ 0 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

Assembling the equivalent nodal load vector $\mathbf{P}^{\textcircled{1}}$ with the aid of the element locating vector $\boldsymbol{\xi}^{\textcircled{1}} = (1 \ 2 \ 3 \ 0 \ 0 \ 0)^{\text{T}}$ into the global nodal load vector $\mathbf{P}$ gives

$$\mathbf{P} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(5) Solve the basic equation to achieve the nodal displacements.

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{F}$$

So, we have

$$\mathbf{A} = \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 824 \\ 5.13 \\ 96.5 \end{bmatrix}$$

The all nodal displacement components are obtained by adding zero displacements

$$\begin{bmatrix} u_A \\ v_A \\ \theta_A \\ u_B \\ v_B \\ \theta_B \\ u_C \\ v_C \\ \theta_C \\ u_D \\ v_D \\ \theta_D \end{bmatrix} = \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 824 \\ 5.13 \\ 96.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(6) Solve the member end force vectors $\bar{\mathbf{F}}^e$.

For element ①, the nodal displacement vector is given by

$$\mathbf{A}^{\textcircled{1}} = \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, the member end force vector under the local coordinate system is formulated

$$\begin{aligned} \bar{\mathbf{F}}^{\textcircled{1}} &= \mathbf{T}\mathbf{k}^{\textcircled{1}}\mathbf{A}^{\textcircled{1}} + \bar{\mathbf{F}}_{\text{P}}^{\textcircled{1}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^{-3} \\ &\times \begin{bmatrix} 2.31 & 0 & -6.94 & -2.31 & 0 & -6.94 \\ 0 & 83.3 & 0 & 0 & -83.3 & 0 \\ -6.94 & 0 & 27.8 & 6.94 & 0 & 13.9 \\ -2.31 & 0 & 6.94 & 2.31 & 0 & 6.94 \\ 0 & -83.3 & 0 & 0 & 83.3 & 0 \\ -6.94 & 0 & 13.9 & 6.94 & 0 & 27.8 \end{bmatrix} \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -0.43 \\ 1.24 \\ -2.09 \\ 0.43 \\ 4.76 \\ -8.49 \end{bmatrix} \end{aligned}$$

For element ②, the nodal displacement vector is specified by

$$\Delta^{\textcircled{2}} = \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 824 \\ 5.13 \\ 96.5 \end{bmatrix}$$

And the member end force vector under the local coordinate system is given by

$$\bar{\mathbf{F}}^{\textcircled{2}} = \mathbf{F}^{\textcircled{2}} = \mathbf{k}^{\textcircled{2}} \Delta^{\textcircled{2}} = 10^{-3} \times \begin{bmatrix} 52.5 & 0 & 0 & -52.5 & 0 & 0 \\ 0 & 0.58 & 3.47 & 0 & -0.58 & 3.47 \\ 0 & 3.47 & 27.8 & 0 & -3.47 & 13.9 \\ -52.5 & 0 & 0 & 52.5 & 0 & 0 \\ 0 & -0.58 & -3.47 & 0 & 0.58 & -3.47 \\ 0 & 3.47 & 13.9 & 0 & -3.47 & 27.8 \end{bmatrix} \begin{bmatrix} 847 \\ -5.13 \\ 28.4 \\ 824 \\ 5.13 \\ 96.5 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 0.43 \\ 2.09 \\ -1.24 \\ -0.43 \\ 3.04 \end{bmatrix}$$

For element ③, the nodal displacement vector is specified by

$$\Delta^{\textcircled{3}} = \begin{bmatrix} 824 \\ 5.13 \\ 96.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And the member end force vector under the local coordinate system is

$$\bar{\mathbf{F}}^{\textcircled{3}} = \mathbf{T} \mathbf{k}^{\textcircled{3}} \Delta^{\textcircled{3}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^{-3} \times \begin{bmatrix} 2.31 & 0 & -6.94 & -2.31 & 0 & -6.94 \\ 0 & 83.3 & 0 & 0 & -83.3 & 0 \\ -6.94 & 0 & 27.8 & 6.94 & 0 & 13.9 \\ -2.31 & 0 & 6.94 & 2.31 & 0 & 6.94 \\ 0 & -83.3 & 0 & 0 & 83.3 & 0 \\ -6.94 & 0 & 13.9 & 6.94 & 0 & 27.8 \end{bmatrix} \begin{bmatrix} 824 \\ 5.13 \\ 96.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.43 \\ -1.24 \\ -3.04 \\ -0.43 \\ 1.24 \\ -4.38 \end{bmatrix}$$

- (7) The internal force diagrams of frame can be drawn according to the member end forces, as shown in figure 8.14.

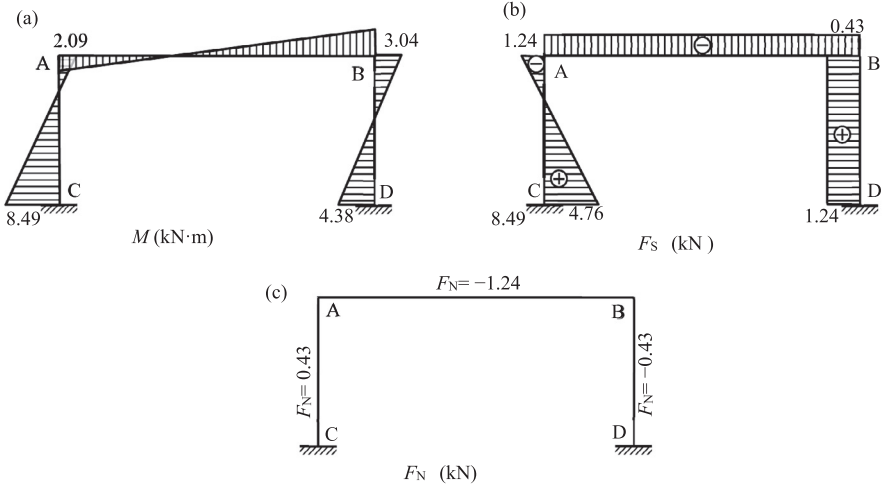


FIG. 8.14 – Internal force diagrams of frame structure.

8.6.3 Example of Composite Structure

For a composite structure with both beam and truss, usually, the elemental stiffness matrix of the axial force bar is utilized for truss analysis, and the elemental stiffness matrix of a general element is applied for beam analysis.

Example 8.5. Calculate the internal forces of the composite structure shown in figure 8.15.

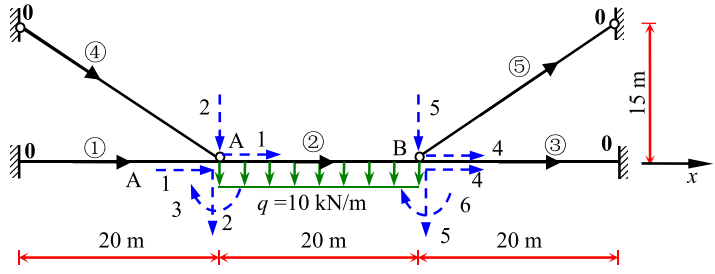


FIG. 8.15 – A composite structure under specific loads and element and node numberings.

Solution: (1) Conduct node and element numberings.

The elastic modulus of the material and the moment of inertia of the cross-section are E and I , respectively. The bending stiffness of the cross-section of beam is EI , and its tension stiffness is $2EI/m^2$. The tension stiffness of the cross-section of the tie rod is $EA = EI/(20 \text{ m}^2)$. The element and node numberings

as well as the global coordinate system are demonstrated in figure 8.12. The element ② is subjected to a uniformly distributed load with a magnitude of 10 kN/m.

(2) Form elemental stiffness matrices.

Elements ①, ②, and ③ are all beam-type members, and thus the elemental stiffness matrices under the local coordinate system are given by

$$\bar{\mathbf{k}}^{\textcircled{1}} = \bar{\mathbf{k}}^{\textcircled{2}} = \bar{\mathbf{k}}^{\textcircled{3}} = \frac{EI}{20} \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.03 & 0.3 & 0 & -0.03 & 0.3 \\ 0 & 0.3 & 4 & 0 & -0.3 & 2 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.03 & -0.3 & 0 & 0.03 & -0.3 \\ 0 & 0.3 & 2 & 0 & -0.3 & 4 \end{bmatrix}$$

For elements ④ and ⑤, according to the elemental stiffness matrix of axial force bar, we have

$$\bar{\mathbf{k}}^{\textcircled{4}} = \bar{\mathbf{k}}^{\textcircled{5}} = EA \begin{bmatrix} 0.04 & 0 & -0.04 & 0 \\ 0 & 0 & 0 & 0 \\ -0.04 & 0 & 0.04 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subsequently, the elemental stiffness matrix is transformed into the global coordinate system. For elements ①, ②, and ③, the angle of two coordinate systems is $\alpha = 0$ and hence the elemental stiffness matrices under global and local coordinate systems are identical.

$$\mathbf{k}^{\textcircled{1}} = \mathbf{k}^{\textcircled{2}} = \mathbf{k}^{\textcircled{3}} = \bar{\mathbf{k}}^{\textcircled{1}} = \bar{\mathbf{k}}^{\textcircled{2}} = \bar{\mathbf{k}}^{\textcircled{3}}$$

For element ④, through computation it is known $\cos \alpha = 0.8$, $\sin \alpha = 0.6$, and the coordinate transformation matrix is given by

$$\mathbf{T} = \begin{bmatrix} 0.8 & 0.6 & | & 0 & 0 \\ -0.6 & 0.8 & | & 0 & 0 \\ \hline 0 & 0 & | & 0.8 & 0.6 \\ 0 & 0 & | & -0.6 & 0.8 \end{bmatrix}$$

And the elemental stiffness matrix under the global coordinate system is computed as

$$\mathbf{k}^{\textcircled{4}} = \mathbf{T}^T \bar{\mathbf{k}}^{\textcircled{4}} \mathbf{T} = EA \begin{bmatrix} 0.0256 & 0.0192 & -0.0256 & -0.0192 \\ 0.0192 & 0.0144 & -0.0192 & -0.0144 \\ -0.0256 & -0.0192 & 0.0256 & 0.0192 \\ -0.0192 & -0.0144 & 0.0192 & 0.0144 \end{bmatrix}$$

For element ⑤, it is known $\cos \alpha = 0.8$, $\sin \alpha = -0.6$ and the coordinate transformation matrix is given by

$$\mathbf{T} = \left[\begin{array}{cc|cc} 0.8 & -0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ \hline 0 & 0 & 0.8 & -0.6 \\ 0 & 0 & 0.6 & 0.8 \end{array} \right]$$

And the elemental stiffness matrix under the global coordinate system is written as

$$\mathbf{k}^{(5)} = \mathbf{T}^T \bar{\mathbf{k}}^{(5)} \mathbf{T} = EA \left[\begin{array}{cccc} 0.0256 & -0.0192 & -0.0256 & 0.0192 \\ -0.0192 & 0.0144 & 0.0192 & -0.0144 \\ -0.0256 & 0.0192 & 0.0256 & -0.0192 \\ 0.0192 & -0.0144 & -0.0192 & 0.0144 \end{array} \right]$$

(3) Assemble the global stiffness matrix with the direct stiffness method.

The element locating vectors of the members shown in figure 8.15 are expressed respectively

$$\xi^{(1)} = [0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3]^T$$

$$\xi^{(2)} = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6]^T$$

$$\xi^{(3)} = [4 \quad 5 \quad 6 \quad 0 \quad 0 \quad 0]^T$$

$$\xi^{(4)} = [0 \quad 0 \quad 1 \quad 2]^T$$

$$\xi^{(5)} = [4 \quad 5 \quad 0 \quad 0]^T$$

Assembling the elemental stiffness matrix with the aid of element locating vector gives the global stiffness matrix ($EA = EI/20$):

$$\mathbf{K} = \frac{EI}{20} \left[\begin{array}{ccc|ccc} 4.0256 & 0.0192 & 0 & -2 & 0 & 0 \\ 0.0192 & 0.0744 & 0 & 0 & -0.03 & 0.3 \\ 0 & 0 & 8 & 0 & -0.3 & 2 \\ \hline -2 & 0 & 0 & 4.0256 & -0.0192 & 0 \\ 0 & -0.03 & -0.3 & -0.0192 & 0.0744 & 0 \\ 0 & 0.3 & 2 & 0 & 0 & 8 \end{array} \right]$$

(4) Form equivalent nodal force vector $\mathbf{P}$.

Only element ② is subjected to external force, and the angle of two coordinate systems is $\alpha = 0$. As a result, the equivalent nodal load vectors under the two coordinate systems are identical.

$$\mathbf{P} = \mathbf{P}^{\textcircled{2}} = - \begin{bmatrix} 0 \\ -\frac{200}{2} \\ -\frac{10}{12} \times 400 \\ 0 \\ -\frac{200}{2} \\ \frac{10}{12} \times 400 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 333 \\ 0 \\ 100 \\ -333 \end{bmatrix}$$

(5) Solve the basic equation.

Solving the following linear algebraic equation set

$$\frac{EI}{20} \begin{bmatrix} 4.0256 & 0.0192 & 0 & -2 & 0 & 0 \\ 0.0192 & 4.0196 & 0 & 0 & -0.03 & 0.3 \\ 0 & 0 & 8 & 0 & -0.3 & 2 \\ -2 & 0 & 0 & 4.0256 & -0.0192 & 0 \\ 0 & -0.03 & -0.3 & -0.0192 & 0.0744 & 0 \\ 0 & 0.3 & 2 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} u_A \\ v_A \\ \theta_A \\ u_B \\ v_B \\ \theta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 333 \\ 0 \\ 100 \\ -333 \end{bmatrix}$$

gives the nodal displacements:

$$\begin{bmatrix} u_A \\ v_A \\ \theta_A \\ u_B \\ v_B \\ \theta_B \end{bmatrix} = \frac{20}{EI} \begin{bmatrix} -12.67 \\ 3976 \\ 2543 \\ 12.67 \\ 3976 \\ -2543 \end{bmatrix}$$

(6) Calculate the end force vectors $\bar{\mathbf{F}}^e$ of members.

$$\begin{aligned} \bar{\mathbf{F}}^{\textcircled{1}} &= \frac{EI}{20} \times \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.03 & 0.3 & 0 & -0.03 & 0.3 \\ 0 & 0.3 & 4 & 0 & -0.3 & 2 \\ \hline -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.03 & -0.3 & 0 & 0.03 & -0.3 \\ 0 & 0.3 & 2 & 0 & -0.3 & 4 \end{bmatrix} \times \frac{20}{EI} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ - \\ -12.67 \\ 3.976 \\ 254.3 \end{bmatrix} \\ &= \begin{bmatrix} 25.34 \\ -42.99 \\ -684.2 \\ - \\ -25.34 \\ 42.99 \\ -175.6 \end{bmatrix} \end{aligned}$$

$$\bar{\mathbf{F}}^{\textcircled{2}} = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.03 & 0.3 & 0 & -0.03 & 0.3 \\ 0 & 0.3 & 4 & 0 & -0.3 & 2 \\ \hline -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.03 & -0.3 & 0 & 0.03 & -0.3 \\ 0 & 0.3 & 2 & 0 & -0.3 & 4 \end{array} \right] \left[\begin{array}{c} -12.67 \\ 3.976 \\ 254.3 \\ - \\ - \\ -254.3 \end{array} \right] + \left[\begin{array}{c} 0 \\ -100 \\ -333 \\ - \\ - \\ 333 \end{array} \right]$$

$$= \left[\begin{array}{c} -50.68 \\ -100 \\ 175.6 \\ - \\ - \\ -175.6 \end{array} \right]$$

$$\bar{\mathbf{F}}^{\textcircled{4}} = \bar{\mathbf{k}}^{\textcircled{4}} \mathbf{T}^{\textcircled{4}} \mathbf{A}^{\textcircled{4}} = \frac{EI}{20} \times \left[\begin{array}{cc|cc} 0.04 & 0 & -0.04 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -0.04 & 0 & 0.04 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cc|cc} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ \hline 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{array} \right]$$

$$\times \left[\begin{array}{c} 0 \\ 0 \\ - \\ -12.67 \\ 3.976 \end{array} \right] \times \frac{20}{EI} = \left[\begin{array}{c} -95.02 \\ 0 \\ - \\ 95.02 \\ 0 \end{array} \right]$$

$$\bar{\mathbf{F}}^{\textcircled{5}} = \left[\begin{array}{c} -95.02 \\ 0 \\ - \\ 95.02 \\ 0 \end{array} \right]$$

(7) The internal force diagrams are drawn, as exhibited in figure 8.16.

8.6.4 Matrix Displacement Method for Rectangular Frame Neglecting Axial Deformation

In practical engineering, rectangular frame structures composed of horizontal beams and vertical columns are widely used, and the influence of axial deformation of such structures is very small and thus can be neglected. When ignoring the axial

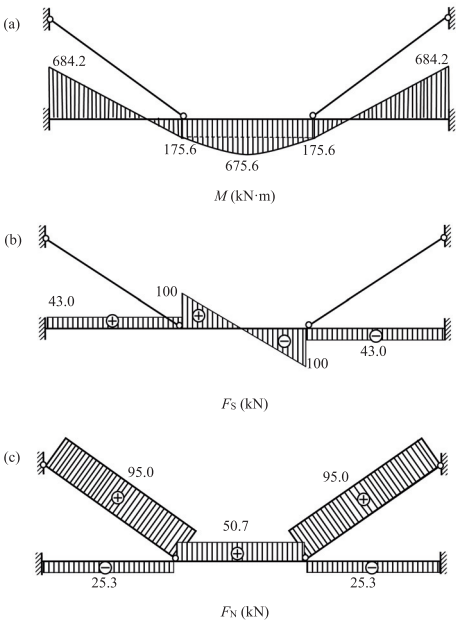


FIG. 8.16 – Internal force diagrams of the composite structure.

deformation and analyzing rectangular frame structures using the matrix displacement method, the following aspects should be noted.

The rectangular frame shown in figure 8.17 is taken as an example for illustration.

Firstly, the global numberings of nodal displacement components are considered. For the fixed supports C and D, all the displacement components are 0 and the global numberings are (0, 0, 0). Due to neglecting the axial deformation, the vertical displacement components at rigid joint A, pinned joints B₁ and B₂ are 0, and thus

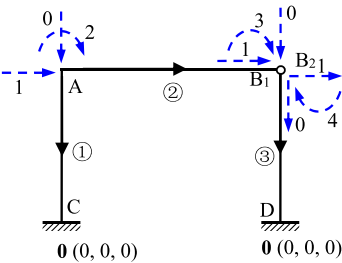


FIG. 8.17 – Element and node numberings of a rectangular frame neglecting axial deformation.

the corresponding global numberings are all 0. Furthermore, since the axial deformation is neglected, the horizontal displacement components of nodes A, B₁, and B₂ are equal to each other, and hence the corresponding global numberings should be identical. The global numberings of all the nodal displacement components are denoted in figure 8.17.

Secondly, when the global stiffness matrix is established by using the matrix displacement method, the element locating vectors are represented as:

$$\xi^{(1)} = [1 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0]^T$$

$$\xi^{(2)} = [1 \quad 0 \quad 2 \quad 1 \quad 0 \quad 3]^T$$

$$\xi^{(3)} = [1 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0]^T$$

It is assumed that the geometrical dimensions of members and material property in figure 8.17 are the same as those of the frame structure shown in figure 8.4 for example 8.1. The elemental stiffness matrices of elements ① and ③ under the global coordinate system are given by

$$\mathbf{k}^{(1)} = \mathbf{k}^{(3)} = 10^4 \times \left[\begin{array}{ccc|ccc} 12 & 0 & -30 & -12 & 0 & -30 \\ 0 & 300 & 0 & 0 & -300 & 0 \\ -30 & 0 & 100 & 30 & 0 & 50 \\ \hline -12 & 0 & 30 & 12 & 0 & 30 \\ 0 & -300 & 0 & 0 & 300 & 0 \\ -30 & 0 & 50 & 30 & 0 & 100 \end{array} \right]$$

The stiffness matrix of element ② is

$$\mathbf{k}^{(2)} = 10^4 \times \left[\begin{array}{ccc|ccc} 300 & 0 & 0 & -300 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ \hline -300 & 0 & 0 & 300 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{array} \right]$$

Then, in terms of element locating vector all the elemental stiffness matrices are assembled into the global stiffness matrix. Since nodal displacements of frame possess 4 non-zero global numberings, the global stiffness matrix is a 4×4 square matrix.

For element ①, the contribution matrix to the global elemental stiffness is written as

$$\begin{array}{c}
 \begin{array}{cc}
 (1) & (3) \\
 \downarrow & \downarrow \\
 1 & 2 & 3 & 4
 \end{array} \\
 \mathbf{K}^{\textcircled{1}} = \begin{array}{l} (1) \rightarrow 1 \\ (3) \rightarrow 2 \\ \rightarrow 3 \\ \rightarrow 4 \end{array} \begin{bmatrix} 12 & -30 & \otimes & \otimes \\ -30 & 100 & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \end{bmatrix} \times 10^4
 \end{array}$$

For element ②, the contribution matrix reads

$$\begin{array}{c}
 \begin{array}{ccc}
 (1), (4) & (3) & (6) \\
 \downarrow & \downarrow & \downarrow \\
 1 & 2 & 3 & 4
 \end{array} \\
 \mathbf{K}^{\textcircled{2}} = \begin{array}{l} (1), (4) \rightarrow 1 \\ (3) \rightarrow 2 \\ (6) \rightarrow 3 \\ \rightarrow 4 \end{array} \begin{bmatrix} 300 - 300 - 300 + 300 & 0 + 0 & 0 + 0 & \otimes \\ 0 + 0 & 100 & 50 & \otimes \\ 0 + 0 & 50 & 100 & \otimes \\ \otimes & \otimes & \otimes & \otimes \end{bmatrix} \times 10^4
 \end{array}$$

For element ③, the contribution matrix is given by

$$\begin{array}{c}
 \begin{array}{ccc}
 (1) & & (3) \\
 \downarrow & & \downarrow \\
 1 & 2 & 3 & 4
 \end{array} \\
 \mathbf{K}^{\textcircled{3}} = \begin{array}{l} (1) \rightarrow 1 \\ \rightarrow 2 \\ \rightarrow 3 \\ (3) \rightarrow 4 \end{array} \begin{bmatrix} 12 & \otimes & \otimes & -30 \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ -30 & \otimes & \otimes & 100 \end{bmatrix} \times 10^4
 \end{array}$$

Adding the contribution matrices from the 3 elements yields the global stiffness matrix of structure

$$\mathbf{K} = \mathbf{K}^{\textcircled{1}} + \mathbf{K}^{\textcircled{2}} + \mathbf{K}^{\textcircled{3}} = \begin{bmatrix} 24 & -30 & 0 & -30 \\ -30 & 200 & 50 & 0 \\ 0 & 50 & 100 & 0 \\ -30 & 0 & 0 & 100 \end{bmatrix} \times 10^4$$

Finally, the basic equation is solved to attain the nodal displacements, and then the member end forces of frame can be calculated and the moment diagram, shear diagram and axial force diagram are drawn.

Problems

8.1 Write the elemental stiffness matrices of the continuous beam.

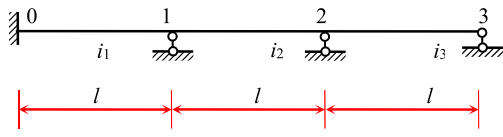


FIG. P8.1

8.2 Calculate the equivalent nodal load vector of node 3 of the frame.

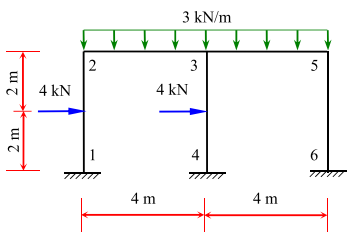


FIG. P8.2

8.3 Compute the equivalent nodal load vector of node 2 for the frame.

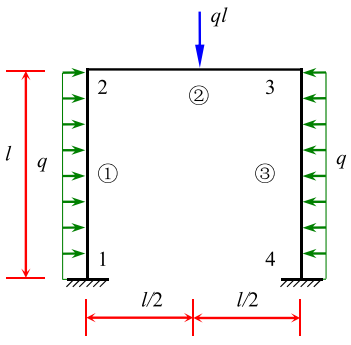


FIG. P8.3

8.4 Calculate the global stiffness matrix of the continuous beam shown in the figure, and solve the equivalent nodal load vector of node 2.

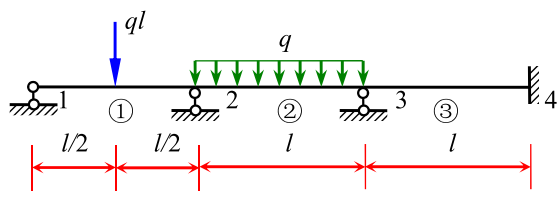


FIG. P8.4

8.5 Calculate the global stiffness matrix of the continuous beam and all the nodal load vectors.

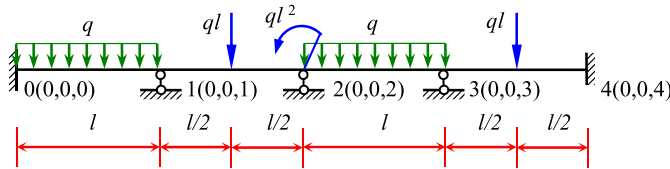


FIG. P8.5

8.6 It is known that the nodal displacement vector (the displacements along x and y directions of nodes 2 and 4) of the truss is: $(1/(EA)) \times [342.322 - 1139.555 - 137.680 - 1167.111]^T$, and EA of each member is a constant. Calculate the internal force of element 1.

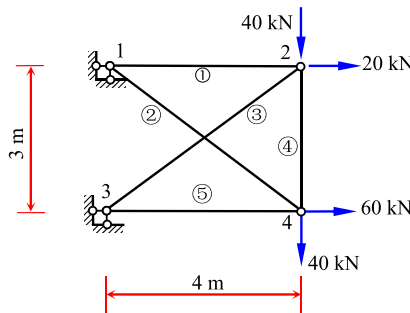


FIG. P8.6

8.7 The elemental stiffness matrix of truss bar 1 is given in equation (P8.7a) and the nodal displacement vector is given in equation (P8.7b). Calculate the axial force of element 1 (tensile or compressive force should be marked).

$$\mathbf{k}_1 = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{P8.7a})$$

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \frac{pl}{EA} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \quad (\text{P8.7b})$$

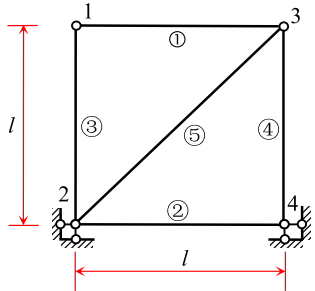


FIG. P8.7

8.8 The material property and cross-section area of each truss bar are $E = 2.1 \times 10^4 \text{ kN/m}^2$, $A = 10^{-2} \text{ m}^2$, and nodal displacement vector is $[0.09524 \ -0.25689]^T$. Compute the nodal forces of element 1.

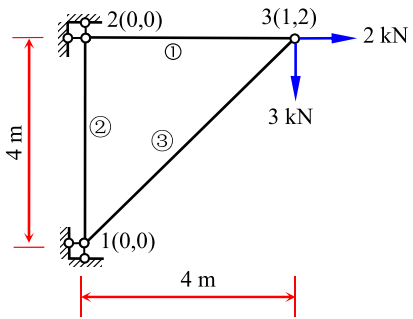


FIG. P8.8

8.9 Calculate the global stiffness matrix $\mathbf{K}$ of the frame structure in the figure. All the members have the same size, $l = 5 \text{ m}$, $A = 0.5 \text{ m}^2$, $I = 1/24 \text{ m}^4$, $E = 3 \times 10^4 \text{ MPa}$.

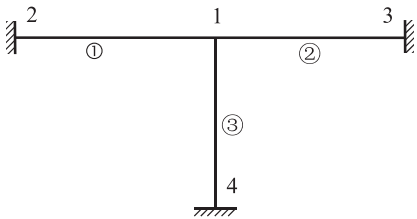


FIG. P8.9

8.10 Write the basic equation of matrix displacement method for the frame structure shown in the figure, and calculate the internal forces of member ends and draw the internal force diagrams. The parameters E , A and I of each member are constants.

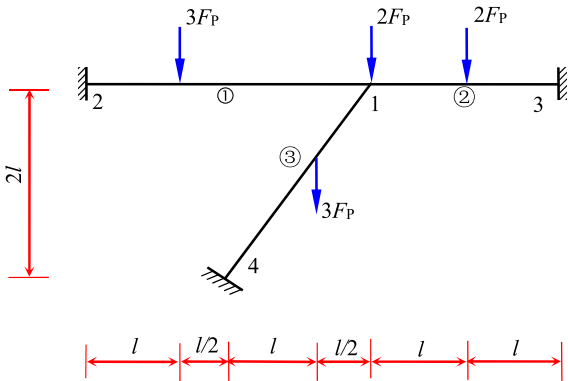


FIG. P8.10

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Appendix A1

Answers to Problems

Chapter 2

- 2.1** (a) Geometrically unchangeable system without redundant constraint
(b) Instantaneously changeable system
(c) Geometrically unchangeable system without redundant constraint
(d) Geometrically unchangeable system without redundant constraint
(e) Geometrically unchangeable system with one redundant constraint
(f) Constantly changeable system
(g) Constantly changeable system
(h) Constantly changeable system
- 2.2** (a) Geometrically unchangeable system with one redundant constraint
(b) Geometrically unchangeable system without redundant constraint
(c) Geometrically unchangeable system without redundant constraint
(d) Geometrically unchangeable system with one redundant constraint
(e) Geometrically unchangeable system without redundant constraint
- 2.3** (a) Geometrically unchangeable system without redundant constraint
(b) Instantaneously changeable system
(c) Geometrically unchangeable system without redundant constraint
(d) Geometrically unchangeable system with one redundant constraint
(e) Constantly changeable system
- 2.4** (a) Geometrically unchangeable system without redundant constraint
(b) Geometrically unchangeable system without redundant constraint
(c) Instantaneously changeable system
- 2.5** (a) Geometrically unchangeable system without redundant constraint
(b) Instantaneously changeable system
(c) Geometrically unchangeable system without redundant constraint
- 2.6** (a) $W = -12$
(b) $W = -3$
- 2.7** (a) $W = -1$
(b) $W = 2$
(c) $W = 0$
(d) $W = 0$

Chapter 3

- 3.1** (a) $M_C = \frac{ql^2}{8}$
 (b) $M_C = \frac{F_P l}{4}$
 (c) $M_C = 10 \text{ kN}\cdot\text{m}$
 (d) $M_C = -10 \text{ kN}\cdot\text{m}$
- 3.2** (a) $M_K = \frac{ql^2}{8}$
 (b) $M_K = \frac{ql^2}{8 \cos \alpha}$
 (c) $M_K = \frac{ql^2}{8 \cos^2 \alpha}$
- 3.3** (a) Right support reaction is 60 kN ($\uparrow$)
 (b) Left support reaction is 25 kN ($\uparrow$)
 (c) Right support moment is $-9Fa$
 (d) Left support moment is $\frac{3}{8}ql^2$
- 3.4** (a) $M_A = 4.50 \text{ kN}\cdot\text{m}$ (tension in bottom fibers), $M_B = 1.00 \text{ kN}\cdot\text{m}$ (tension in top fibers), $M_C = 1.875 \text{ kN}\cdot\text{m}$ (tension in top fibers)
 (b) $M_B = 20 \text{ kN}\cdot\text{m}$ (tension in top fibers), $F_{SC} = 0$
 (c) $M_K = 47.5 \text{ kN}\cdot\text{m}$
- 3.5** $x = \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)l = 0.1464l$.
- 3.6** (a) $F_{Na} = -\frac{5\sqrt{2}}{6}F_P$, $F_{Nb} = \frac{5}{3}F_P$
 (b) $F_{N1} = 10 \text{ kN}$, $F_{N2} = -10 \text{ kN}$, $F_{N3} = 0$
- 3.7** (a) $F_{N1} = -0.25F_P$, $F_{N2} = -2.67F_P$, $F_{N3} = -0.417F_P$
 (b) $F_{N1} = 52.5 \text{ kN}$, $F_{N2} = 18 \text{ kN}$, $F_{N3} = -18 \text{ kN}$
 (c) $F_{N1} = 150 \text{ kN}$, $F_{N2} = -32.3 \text{ kN}$, $F_{N3} = -124.2 \text{ kN}$
 (d) $F_{N1} = 3.75 \text{ kN}$, $F_{N2} = 12.5 \text{ kN}$, $F_{N3} = -11.25 \text{ kN}$
- 3.8** (a) $F_{NAB} = -4 \text{ kN}$
 (b) $F_{xA} = 0.25F_P$ ($\leftarrow$), $F_{NAB} = 1.35F_P$
- 3.10** (a) $M_B = qa^2$ (tension in inside fibers)
 (b) $M_A = M_B = 1.125 \text{ kN}\cdot\text{m}$ (tension in inside fibers),
 $M_C = 12.375 \text{ kN}\cdot\text{m}$ (tension in bottom fibers)
 (c) $M_D = 2.32 \text{ kN}\cdot\text{m}$ (tension in inside fibers)
 (d) $M_D = 3.962 \text{ kN}\cdot\text{m}$ (tension in inside fibers)
 (e) $F_{yA} = qa$ ($\downarrow$), $F_{xA} = 3qa$ ($\leftarrow$), $F_{yB} = qa$ ($\uparrow$), $F_{xB} = qa$ ($\rightarrow$)
 (f) $M_{DE} = 80 \text{ kN}\cdot\text{m}$ (tension in right fibers), $M_{BA} = -80 \text{ kN}\cdot\text{m}$ (tension in left fibers)
 (g) $M_F = 180 \text{ kN}\cdot\text{m}$ (tension in outside fibers), $M_G = 67.5 \text{ kN}\cdot\text{m}$ (tension in outside fibers), $M_H = 652.5 \text{ kN}\cdot\text{m}$ (tension in outside fibers)
- 3.11** (a) $F_{NAD} = -12 \text{ kN}$, $F_{SAC} = 3 \text{ kN}$
 (b) $F_{NFG} = 358 \text{ kN}$, $F_{NFD} = -81.9 \text{ kN}$, $M_{DC} = 14.7 \text{ kN}\cdot\text{m}$ (tension in bottom fibers)

$$\mathbf{3.12} \quad (\text{a}) \quad F_{yA} = \frac{3}{4}F_P, \quad F_{yB} = \frac{1}{4}F_P, \quad F_H = \frac{1}{2}F_P$$

$$(\text{b}) \quad M_E = -0.5 \text{ m} \cdot F_P$$

$$(\text{c}) \quad F_{ND}^L = -0.78F_P, \quad F_{SD}^L = 0.45F_P$$

$$\mathbf{3.13} \quad (\text{a}) \quad F_{yA}, F_{yB} \text{ remain unchanged, } F_H \text{ reduces by half, } M \text{ remains unchanged}$$

$$(\text{b}) \quad \text{The reactions remain unchanged, arch height and span double their values, and } M \text{ also doubles its value.}$$

Chapter 4

$$\mathbf{4.1} \quad \Delta_{xB} = \frac{2(1 + \sqrt{2})F_P a}{EA} (\rightarrow)$$

$$\mathbf{4.2} \quad \Delta_{yC} = 0$$

$$\mathbf{4.3} \quad \Delta_{AB} = \frac{\sqrt{2}Pa}{EA}$$

$$\mathbf{4.4} \quad \Delta_{yB} = 1.44 \times 10^{-3} \text{ m} (\downarrow), \quad \Delta_{yD} = 0.3 \times 10^{-3} \text{ m} (\downarrow)$$

$$\mathbf{4.5} \quad \Delta_y = \frac{F_P l^3}{48EI} (\downarrow)$$

$$\mathbf{4.6} \quad \Delta_C = \Delta_M + \Delta_S = \frac{5ql^4}{384EI} + \frac{kql^2}{8GA} (\downarrow); \text{ when } \frac{h}{l} = \frac{1}{10}, \quad \frac{\Delta_S}{\Delta_M} = 2.56\%$$

$$\mathbf{4.7} \quad \Delta_{yB} = \frac{qR^4}{2EI} \left(\frac{2}{3} - \cos \alpha + \frac{1}{3} \cos^3 \alpha \right) (\downarrow)$$

$$\mathbf{4.8} \quad \Delta_{xB} = \frac{ql^3}{15EI} (\rightarrow)$$

$$\mathbf{4.9} \quad f_{\max} = \frac{23F_P l^3}{648EI} (\downarrow)$$

$$\mathbf{4.10} \quad \Delta_{yC} = 3.23 \text{ mm} (\downarrow)$$

$$\mathbf{4.11} \quad \Delta_{xB} = \frac{432q}{EI_1} (\rightarrow)$$

$$\mathbf{4.12} \quad \Delta_{yC} = \frac{27ql^4}{16EI} (\downarrow)$$

$$\mathbf{4.13} \quad \Delta_{yD} = \frac{229ql^4}{384EI} (\downarrow)$$

$$\mathbf{4.14} \quad \Delta_{yC} = \frac{11ql^4}{48EI} (\downarrow)$$

$$\mathbf{4.15} \quad \Delta_{yC} = -\frac{3}{8} \alpha t a (\uparrow)$$

$$\mathbf{4.16} \quad \Delta_{yC} = -180\alpha - \frac{1080\alpha}{h} (\uparrow)$$

$$\mathbf{4.17} \quad \Delta_{yC} = -\frac{179\alpha t l}{2} (\uparrow)$$

$$\mathbf{4.18} \quad \theta = \frac{Ml}{2EI} - \frac{\alpha t_1 l}{h}; \text{ if } \theta = 0, \text{ then } M = \frac{2EI\alpha t_1}{h}$$

$$\mathbf{4.19} \quad \Delta = \frac{5ql^4}{384EI} (\downarrow)$$

Chapter 5

- 5.1** (a) 7 degrees
 (b) 3 degrees
 (c) 3 degrees
 (d) 21 degrees
- 5.2** (a) $F_{yC} = \frac{5}{8} F_P$
 (b) $M_{AB} = 7.2 \text{ kN}\cdot\text{m}$ (tension in left fibers)
- 5.3** (a) $M_A = 2.94 \text{ m} \cdot F_P$, $M_B = 4.06 \text{ m} \cdot F_P$ (tension in left fibers)
 (b) $F_{NDE} = -17.39 \text{ kN}$ (Compression), $M_{AD} = 248.5 \text{ kN}\cdot\text{m}$ (tension in left fibers)
- 5.4** (a) $F_{RB} = 1.172 F_P$ ($\uparrow$)
 (b) $F_{NBC} = 0.896 F_P$
- 5.5** The tension of the central bar is 7.66 kN, and the moment at the central hanging point is 4.97 kN·m (tension in top fibers)
- 5.6** (a) $M_{AB} = \frac{1}{6} qa^2$ (tension in outside fibers)
 (b) $M_{EC} = 1.8F_P$ (tension in inside fibers), $M_{CE} = 1.2F_P$ (tension in outside fibers), $M_{CA} = 3F_P$ (tension in inside fibers), $M_{CD} = 4.2F_P$ (tension in bottom fibers)
- 5.7** $k = \frac{24EI}{l^3}$, $\Delta_B = \frac{ql^4}{24EI}$
- 5.8** $F_H = \frac{F_P}{\pi}$
- 5.9** $F_H = \frac{ql^2}{8f} \frac{1}{1 + \frac{15}{8} \frac{EI}{E_1 A_1 f^2}}$
- 5.10** $M_{AC} = \frac{9\Delta}{268a^2} EI$ (tension in outside fibers)
- 5.11** (a) $M_{AB} = \frac{3EI}{l} \alpha$ (tension in bottom fibers), $y(x) = \frac{\alpha}{2l^2} (2l^2 x + 3lx^2 - x^3)$,
 $y_{\max} = 0.192\alpha l$
 (b) $M_{AB} = \frac{4EI}{l} \alpha$ (tension in bottom fibers), $y(x) = \frac{\alpha}{l^2} (l^2 x - 2lx^2 + x^3)$,
 $y_{\max} = 0.148\alpha l$
- 5.12** $M_{AB} = \frac{6EI}{l^2} c$ (tension in top fibers)
- 5.13** $M_{AB} = \frac{3EI}{2h} \alpha(t_2 - t_1)$, $y(x) = \frac{\alpha(t_2 - t_1)}{4h} \left(x^2 - \frac{x^3}{l} \right)$
- 5.14** Figure (a) when $f = 2 \text{ m}$, $s = 0.2$, then $M_C = 17.34 \text{ kN}\cdot\text{m}$ (tension in inside fibers), $M_D = 48.44 \text{ kN}\cdot\text{m}$ (tension in outside fibers)

Chapter 6

- 6.1** (a) $M_{DB} = 4i\theta_D$, $M_{DA} = 3i\theta_D + \frac{3ql^2}{16}$, $M_{DC} = i\theta_D - \frac{1}{3}ql^2$, $8i\theta_D - \frac{7}{48}ql^2 = 0$
 (b) $M_{DA} = 3i\theta_D + \frac{1}{8}ql^2$, $M_{DB} = 4i\theta_D$, $M_{DC} = -ql^2$, $i\theta_D = \frac{1}{8}ql^2$
 (c) $M_{EA} = 3\frac{E}{l}\theta_E$, $M_{ED} = 24\frac{E}{l}\theta_E$, $M_{EC} = 2\frac{E}{l}\theta_E$, $M_{EB} = 0$, $29\frac{E}{l}\theta_E = M_0$
 (d) $M_{AB} = 4i\theta_A + 2i\theta_B$, $M_{AD} = 4i\theta_A$, $M_{AF} = 3i\theta_A - \frac{3ql^2}{16}$, $M_{BC} = i\theta_B + \frac{ql^2}{3}$,
 $15i\theta_A + 2i\theta_B - \frac{3ql^2}{16} = 0$, $2i\theta_A + 5i\theta_B + \frac{ql^2}{3} = 0$
 (e) $M_{BA} = 2i\theta_A + 4i\theta_B$, $M_{BC} = 3i\theta_B - 3\frac{i}{l}A_C$, $M_{CD} = 0$,
 $2i\theta_A + 7i\theta_B - 3\frac{i}{l}A_C = 0$
- 6.2** $F_{N1} = -50 \text{ kN}$, $F_{N2} = -75 \text{ kN}$, $F_{N3} = -100 \text{ kN}$, $F_{N4} = -125 \text{ kN}$,
 $F_{N5} = -150 \text{ kN}$
- 6.3** $M_{AB} = 27.2 \text{ kN}\cdot\text{m}$, $M_{BC} = -54.3 \text{ kN}\cdot\text{m}$, $M_{CB} = 70.3 \text{ kN}\cdot\text{m}$
- 6.4** $M_{AC} = -225 \text{ kN}\cdot\text{m}$, $M_{BD} = -135 \text{ kN}\cdot\text{m}$, $F_{SAC} = 97.5 \text{ kN}$
- 6.5** $M_{AC} = -34.49 \text{ kN}\cdot\text{m}$, $M_{CA} = 14.58 \text{ kN}\cdot\text{m}$, $M_{BD} = -20.09 \text{ kN}\cdot\text{m}$
- 6.6** $M_{AC} = -8.43 \text{ kN}\cdot\text{m}$, $M_{CD} = 2.07 \text{ kN}\cdot\text{m}$, $M_{DC} = 3.07 \text{ kN}\cdot\text{m}$,
 $F_{SAC} = 4.75 \text{ kN}$, $F_{SCA} = -1.25 \text{ kN}$, $F_{SCD} = -0.43 \text{ kN}$,
 $F_{NAC} = 0.43 \text{ kN}$, $F_{NBD} = -0.43 \text{ kN}$, $F_{NCD} = -1.25 \text{ kN}$
- 6.7** $M_{AD} = \frac{1}{48}ql^2$, $M_{DE} = -\frac{1}{24}ql^2$
- 6.8** $M_{AC} = M_{BD} = -171.4 \text{ kN}\cdot\text{m}$, $M_{CA} = M_{DB} = -128.6 \text{ kN}\cdot\text{m}$
- 6.9** $M_{AB} = 59.14 \text{ kN}\cdot\text{m}$, $M_{BA} = -60.86 \text{ kN}\cdot\text{m}$, $M_{CA} = 59.14 \text{ kN}\cdot\text{m}$,
 $M_{DB} = -60.86 \text{ kN}\cdot\text{m}$
- 6.10** $M_{AB} = -1.388 \text{ q}\cdot\text{m}^2$, $M_{AC} = -0.6074 \text{ q}\cdot\text{m}^2$
- 6.11** (a) $M_{BA} = 0$, $M_{BD} = \frac{4}{15}ql^2$, $M_{DB} = \frac{2}{15}ql^2$, $M_{BC} = -\frac{4}{15}ql^2$, $M_{CB} = -\frac{7}{30}ql^2$
 (b) $M_{BA} = \frac{1}{8}ql^2$, $M_{BD} = \frac{1}{6}ql^2$, $M_{DB} = \frac{1}{12}ql^2$, $M_{BC} = -\frac{7}{24}ql^2$, $M_{CB} = -\frac{5}{24}ql^2$
- 6.12** $M_{AC} = \frac{115}{12} \text{ kN}\cdot\text{m}$, $M_{CA} = \frac{115}{6} \text{ kN}\cdot\text{m}$, $M_{CE} = \frac{155}{6} \text{ kN}\cdot\text{m}$, $M_{EC} = \frac{5}{12} \text{ kN}\cdot\text{m}$,
 $M_{CD} = -45 \text{ kN}\cdot\text{m}$
- 6.13** $M_{AB} = 0$, $M_{BA} = 20 \text{ kN}\cdot\text{m}$, $M_{BC} = -40 \text{ kN}\cdot\text{m}$, $M_{BD} = 20 \text{ kN}\cdot\text{m}$,
 $M_{DB} = 10 \text{ kN}\cdot\text{m}$
- 6.14** $M_{AB} = -20 \text{ kN}\cdot\text{m}$, $M_{BA} = 0$, $M_{BD} = 0$, $M_{DB} = -8.18 \text{ kN}\cdot\text{m}$,
 $M_{CD} = -75.45 \text{ kN}\cdot\text{m}$, $M_{DC} = -20.91 \text{ kN}\cdot\text{m}$, $M_{DE} = 29.09 \text{ kN}\cdot\text{m}$,
 $M_{ED} = 34.55 \text{ kN}\cdot\text{m}$
- 6.15** $M_{AB} = -45 \text{ kN}\cdot\text{m}$, $M_{BA} = -30 \text{ kN}\cdot\text{m}$, $M_{BC} = -25 \text{ kN}\cdot\text{m}$, $M_{CB} = 0$
- 6.16** $M_{AB} = -49.4 \text{ kN}\cdot\text{m}$, $M_{BA} = -12.3 \text{ kN}\cdot\text{m}$, $M_{BC} = 12.3 \text{ kN}\cdot\text{m}$,
 $M_{CB} = 20.1 \text{ kN}\cdot\text{m}$, $M_{CD} = -20.1 \text{ kN}\cdot\text{m}$, $M_{DC} = -26.2 \text{ kN}\cdot\text{m}$
- 6.17** $M_{AB} = 0$, $M_{BA} = 38.13 \text{ kN}\cdot\text{m}$, $M_{BC} = -48.44 \text{ kN}\cdot\text{m}$, $M_{CB} = 0$,
 $M_{BD} = 10.30 \text{ kN}\cdot\text{m}$, $M_{DB} = 5.15 \text{ kN}\cdot\text{m}$

$$\begin{aligned}
\mathbf{6.18} \quad M_{AC} &= -\frac{1523Fl}{10000}, M_{CA} = -\frac{1811Fl}{10000}, M_{CD} = \frac{1155Fl}{10000}, M_{DC} = \frac{511Fl}{10000}, \\
M_{DB} &= \frac{770Fl}{10000}, M_{BD} = \frac{896Fl}{10000}, M_{AE} = \frac{1523Fl}{10000}, M_{EA} = \frac{1523Fl}{10000}, \\
M_{EF} &= -\frac{1523Fl}{10000}, M_{FE} = -\frac{1811Fl}{10000}, M_{FC} = \frac{656Fl}{10000}, M_{CF} = \frac{656Fl}{10000}, \\
M_{FG} &= \frac{1155Fl}{10000}, M_{GF} = \frac{511Fl}{10000}, M_{GH} = \frac{770Fl}{10000}, M_{HG} = \frac{896Fl}{10000}, \\
M_{GD} &= -\frac{1281Fl}{10000}, M_{DG} = -\frac{1281Fl}{10000}, M_{HB} = -\frac{896Fl}{10000}, M_{BH} = -\frac{896Fl}{10000} \\
\Delta_{Fy} &= 0.0476 \frac{Fl^3}{EI} \quad (\downarrow)
\end{aligned}$$

Chapter 7

$$\mathbf{7.1} \quad (a) \quad \bar{F}_{yA} = 1, \bar{M}_A = -x,$$

$$\bar{M}_C = \begin{cases} 0 & (0 \leq x \leq a) \\ -(x-a) & (a \leq x \leq l) \end{cases}, \quad \bar{F}_{SC} = \begin{cases} 0 & (0 \leq x \leq a) \\ 1 & (a \leq x \leq l) \end{cases}$$

$$(b) \quad \bar{F}_{yA} = \bar{F}_{yA}^0, \bar{M}_C = \bar{M}_C^0, \bar{F}_{SC} = \bar{F}_{SC}^0 \cos \alpha, \bar{F}_{NC} = -\bar{F}_{SC}^0 \sin \alpha$$

where the superscript "0" represents the influence coefficient of the corresponding horizontal beam

$$(c) \quad \bar{F}_{SA}^L = -1 \text{ (left side of point A)}, \bar{F}_{SA}^R = 1 \text{ (right side of point A)}, \bar{M}_C = 4a \text{ (point C)}$$

$$(d) \quad \bar{F}_{RA} = 1 \text{ (point A)}, \bar{F}_{SB} = -1 \text{ (point B}^L), \bar{M}_E = \frac{a}{2} \text{ (point E)}$$

Note: variables with a bar represent the influence coefficients, the same below.

$$\mathbf{7.2} \quad \bar{F}_{RA} = 1 \text{ (point A)}, \bar{F}_{RA} = 0 \text{ (right side of point B)}, \bar{F}_{RC} = \frac{l+c}{l} \text{ (point B)},$$

$$\bar{F}_{RD} = -\frac{c}{l} \text{ (point B)}, \bar{F}_{SB} = -1 \text{ (point B}^L), \bar{F}_{SB} = 0 \text{ (right side of point B)},$$

$$\bar{M}_E = \frac{ab}{l} \text{ (point E)}, \bar{M}_F = -\frac{ce}{l} \text{ (point B)}, \bar{F}_{SE} = -\frac{a}{l} \text{ (point E}^L), \bar{F}_{SF} = \frac{c}{l} \text{ (point B)}$$

$$\mathbf{7.3} \quad \bar{M}_A = -l \text{ (point C)}, \bar{F}_{yA} = 1 \text{ (segment BC)}, \bar{M}_K = -a \text{ (point C)}, \bar{F}_{SK} = 1 \text{ (point C}^R)$$

$$\mathbf{7.4} \quad \bar{F}_{RA} = -\frac{1}{4} \text{ (point H)}, \bar{F}_{RC} = \frac{5}{4} \text{ (point I)}, \bar{F}_{SB}^L = -\frac{1}{4} \text{ (point H)}, \bar{F}_{SB}^R = 1 \text{ (point H)},$$

$$\bar{M}_F = -\frac{1}{2} \text{ m (point H)}, \bar{F}_{SF} = -\frac{1}{4} \text{ (point H)}, \bar{M}_G = \frac{1}{2} \text{ m (point G)}, \bar{F}_{SG} = \frac{1}{2} \text{ (right side of point G)}$$

$$\mathbf{7.5} \quad \bar{F}_{RA} = -1, \bar{F}_{RB} = 2, \bar{M}_A = 2 \text{ m (all for point F)}$$

$$\mathbf{7.6} \quad \text{loading at top chords: } \bar{F}_{N1} = -2 \text{ (point C)}, \bar{F}_{N2} = \frac{2\sqrt{2}}{5} \text{ (point D)}, \bar{F}_{N3} = \frac{12}{5} \text{ (point C)},$$

loading at bottom chords: $\bar{F}_{N1} = -\frac{5}{2}$ (point I), $\bar{F}_{N2} = \frac{\sqrt{2}}{2}$ (point I), $\bar{F}_{N3} = \frac{9}{5}$ (point H)

7.7 $\bar{M}_C = \frac{h}{2}$, $\bar{F}_{SC} = -\frac{h}{l}$ (all for point D)

7.8 $\bar{M}_E = -0.667$ m (point C), $F_{SB}^L = -0.667$ (point C), $F_{SB}^R = 1$ (point C)

7.9 See 7.4 and 7.5

7.10 $\bar{F}_{N1} = \frac{3}{2}$, $\bar{F}_{N2} = 1$, $\bar{F}_{N3} = -\frac{3}{2}$, $\bar{F}_{N4} = \sqrt{2}$ (all for point C)

7.11 $F_{RA} = 5$ kN, $F_{RB} = 55$ kN, $F_{SC} = -5$ kN, $M_C = 0$

7.12 $Z_{\max} = -1555$ kN

7.13 $M_{C\max} = 314$ kN·m, $F_{SC\max} = 104.8$ kN, $F_{SC\min} = -27.3$ kN

7.14 $y(x) = \begin{cases} \frac{x^2}{2l^2} \left(\frac{x}{l} - 3 \right) & (0 \leq x \leq a) \\ \frac{x^2}{2l^2} \left(\frac{x}{l} - 3 \right) + 1 & (a \leq x \leq l) \end{cases}$

7.15

$$y(x) = \begin{cases} \frac{13367}{3677441600} x(x-80)(x+80) & (0 \leq x \leq 80) \\ -\frac{1}{45968020000} (x-80)(27457x^2 - 215591720x + 14932892800) & (80 \leq x \leq 130) \\ -\frac{1}{45968020000} (x-180)(27457x^2 - 212846020x + 39835966800) & (130 \leq x \leq 180) \\ -\frac{103}{10112964400} (x-180)(x-290)(391x - 117240) & (180 \leq x \leq 290) \\ -\frac{103}{919360400} (x+260)(x-290)(x-360) & (290 \leq x \leq 360) \end{cases}$$

Chapter 8

8.1 $[K] = \begin{bmatrix} 4(i_1 + i_2) & 2i_2 & 0 \\ 2i_2 & 4(i_2 + i_3) & 2i_3 \\ 0 & 2i_3 & 4i_3 \end{bmatrix}$

8.2 $\begin{bmatrix} 2 \text{ kN} \\ -12 \text{ kN} \\ 2 \text{ kN} \cdot \text{m} \end{bmatrix}$

8.3 $\begin{bmatrix} ql/2 \\ -ql/2 \\ -ql^2 \end{bmatrix}$

8.4 $\begin{bmatrix} -ql \\ ql^2/24 \end{bmatrix}$

8.5 $\begin{bmatrix} -ql^2/24 \\ 25ql^2/24 \\ -ql^2/24 \\ ql^2/8 \end{bmatrix}$

8.6 $\begin{bmatrix} -85.581 \text{ kN} \\ 85.585 \text{ kN} \end{bmatrix}$

8.7 $3P$ (Compression)

8.8 $\begin{bmatrix} -5 \text{ kN} \\ 0 \\ 5 \text{ kN} \\ 0 \end{bmatrix}$

8.9 $\mathbf{K} = 10^4 \times \begin{bmatrix} 612 & 0 & -30 \\ 0 & 324 & 0 \\ -30 & 0 & 300 \end{bmatrix}$

8.10 $\begin{bmatrix} 0.977 \frac{EA}{l} + 0.492 \frac{EI}{l^3} & -0.192 \frac{EA}{l} + 0.369 \frac{EI}{l^3} & -0.768 \frac{EI}{l^2} \\ -0.192 \frac{EA}{l} + 0.369 \frac{EI}{l^3} & 0.256 \frac{EA}{l} + 2.221 \frac{EI}{l^3} & 0.257 \frac{EI}{l^2} \\ -0.768 \frac{EI}{l^2} & 0.257 \frac{EI}{l^2} & 4.933 \frac{EI}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0.107 F_P \\ 5.858 F_P \\ -5 F_P l / 6 \end{bmatrix}$

Appendix A2

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Appendix A3

Matlab Program Codes of the Matrix Displacement Method for Plane Structural Analysis

Example 1: A plane truss (example 8.3 in section 8.6.1 in the textbook)

1. The program flowchart

In the following program flowchart in figure A1, the Matlab code of the matrix displacement method for a plane truss in example 8.3 in section 8.6.1 in this textbook is illustrated. Material properties and other basic information are defined in the beginning, and the variables are applied to conduct symbol calculation which can set specific values according to the required operation. For a truss, the

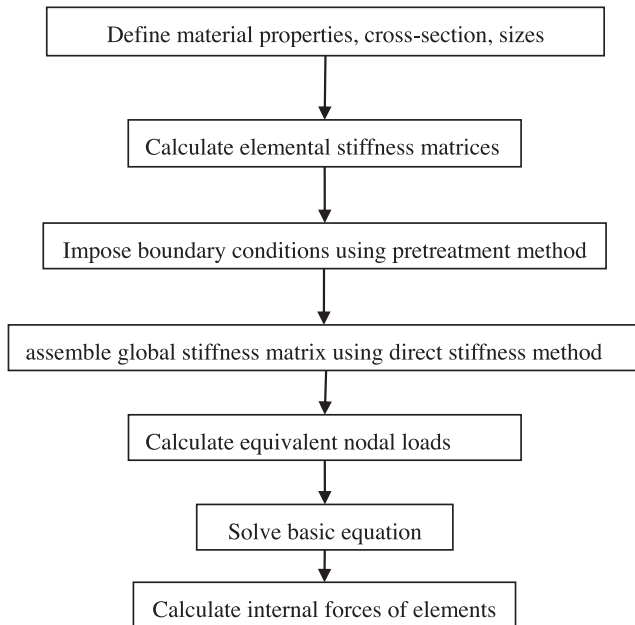


FIG. A1 – Program flowchart of matrix displacement method for a plane truss.

dimension of the elemental stiffness matrix is 4. We utilize the pretreatment method to impose the boundary conditions and apply the direct stiffness method to form the global stiffness matrix. Then, we calculate the equivalent nodal forces and solve the basic equation of the matrix displacement method. After obtaining the nodal displacements of the truss, they are used to calculate the internal force of each bar.

2. Variable definition

TAB. A1 – Variable definition in the Matlab code for analysis of plane truss.

Variable	Definition
E	Elastic modulus of material
A	Cross-section area of bar
L	Length of bar
k1, k2, k3, k4, k5, k6	Elemental stiffness matrices of truss bars
T1, T2, T3, T4, T5, T6	Coordinate transformation matrices
K	Global stiffness matrix
F	Nodal load vector
U	Nodal displacement vector of structure
U1, U2, U3, U4, U5, U6	Nodal displacement vectors of elements
F1, F2, F3, F4, F5, F6	Nodal force vectors of elements

Example 2: A plane frame structure
(example 8.4 in section 8.6.2)

1. The program flowchart

In the program flowchart in figure A2, the Matlab code of the matrix displacement method for a plane frame in example 8.4 in section 8.6.2 is illustrated. Material properties and other basic information are defined in the beginning, and here we use symbol calculation such that the variables can be defined later. In this example, the two columns have the same elemental stiffness matrix, thus we only calculate one of them. The elemental stiffness matrix of the beam should be calculated separately. The boundary conditions are imposed, and the elemental stiffness matrices are assembled to form a global stiffness matrix by the direct stiffness method. Finally, the basic equation of frame structure is solved. After obtaining the nodal displacements, the nodal forces of each element are calculated.

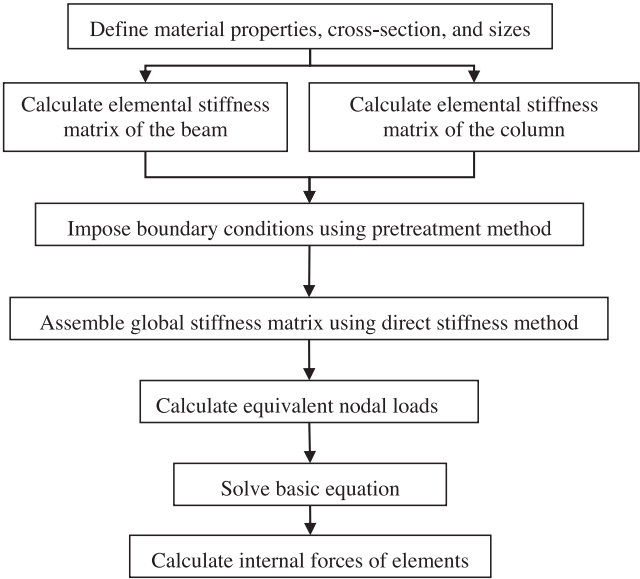


FIG. A2 – Program flowchart of matrix displacement method for a plane frame.

2. Variable Definition

TAB. A2 – Variable definition in the Matlab code for analysis of plane frame.

Variable	Definition
E	Elastic modulus of material
A1, A2	Cross-section areas of the column and beam
l1, l2	Lengths of the column and beam
I1, I2	Inertial moments of the column and beam
k1, k2	Elemental stiffness matrices of the column and beam
T	Coordinate transformation matrix
K	Global stiffness matrix
f	Nodal load vector
U	Nodal displacement vector of structure
f1, f2, f3	Nodal force vector vectors of elements

Example 3: A composite structure
(example 8.5 in section 8.6.3)

1. The program flowchart

In the following flowchart in figure A3, the Matlab code of the matrix displacement method for a composite structure in example 8.5 in section 8.6.3 is illustrated. Material properties and other basic information are defined in the beginning, and we use the variables to conduct symbol calculation which can be set to specific values.

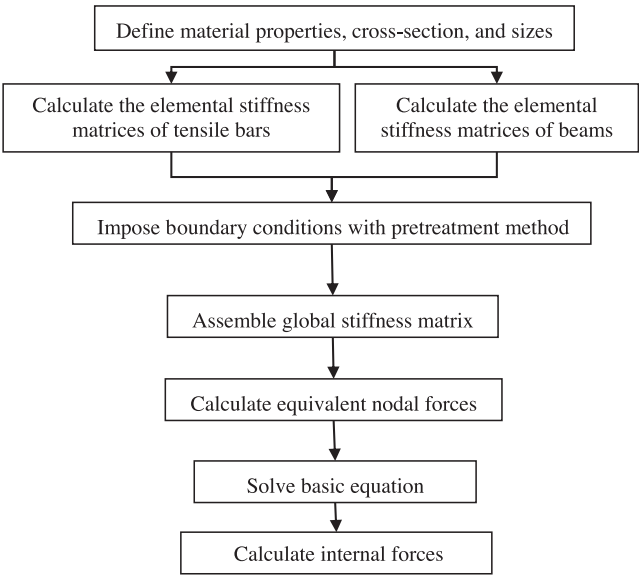


FIG. A3 – Program flowchart of matrix displacement method for a composite structure.

2. Variable Definition

TAB. A3 – Variable definition in the Matlab code of matrix displacement method for a composite structure.

Variable	Definition
E	Elastic modulus of material
A	Cross-section area of member
L	Length of member
k1, k2, k3, k4, k5	Element stiffness matrices of tensile bars and beams
T1, T2, T3, T4, T5	Coordinate transformation matrices
K	Global stiffness matrix
F	Nodal load vector
U	Nodal displacement vector of structure
U1, U2, U3, U4, U5	Nodal displacement vectors of elements

Example 4: A pane frame structure
(example 8.2 in section 8.5.2)

1. The Program Flowchart

In the case of all the element stiffness matrix under the global coordinate system provided, this example focuses on the procedures of assembling of element stiffness matrix and solving the primary equation. The program flowchart of the matrix displacement method for a plane frame in example 8.2 in section 8.5.2 is illustrated as follows.

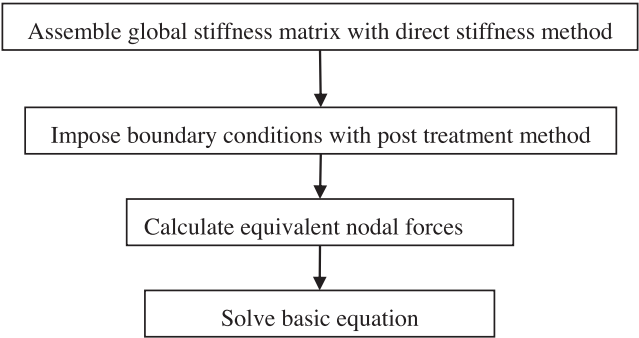


FIG. A4 – Program flowchart of matrix displacement method for frame structure.

2. Variable Definition

TAB. A4 – Variable definition in the Matlab code of matrix displacement method for frame structure.

Variable	Definition
k1, k2	Elemental stiffness matrices under global coordinate system
K	Global stiffness matrix
F	Nodal force vector
U	Nodal displacement vector of frame structure
U1, U2, U3, U4, U5	Nodal displacement vectors of elements

Matlab codes of the four examples

1. Example 1: Plane truss

```
%All rights reserved: Dalian University of Technology
%Matlab implementation of the matrix displacement method
%Solving details:
%-----
syms E A l
k1_loc = E*A/l*[1 0 -1 0; 0 0 0 0; -1 0 1 0; 0 0 0 0];
```

```

k2_loc = k1_loc;
k3_loc = k1_loc;
k4_loc = k1_loc;
k5_loc = E*A/l*[1/sqrt(2) 0 -1/sqrt(2) 0; 0 0 0 0 ;...
               -1/sqrt(2) 0 1/sqrt(2) 0; 0 0 0 0];
k6_loc = k5_loc;
T2 = eye(4);
k2 = T2'*k2_loc*T2;
T4 = T2;
k4 = k2;
T1 = [0 1 0 0; -1 0 0 0; 0 0 0 1; 0 0 -1 0];
k1 = T1'*k1_loc*T1;
T3 = T1;
k3 = k1;
T5 = sqrt(2)/2*[1 1 0 0; -1 1 0 0; 0 0 1 1; 0 0 -1 1];
k5 = T5'*k5_loc*T5;
T6 = sqrt(2)/2*[-1 1 0 0; -1 -1 0 0; 0 0 -1 1; 0 0 -1 -1];
k6 = T6'*k6_loc*T6;
%Form the global stiffness matrix by direct assembling
%The locating vectors of each element are
d1 = [1 2 0 0];
d2 = [1 2 3 4];
d3 = [3 4 0 0];
d4 = [0 0 0 0];
d5 = [1 2 0 0];
d6 = [3 4 0 0];
K = 0*k1;
for i = 1:4
    for j = 1:4
        if(d1(i) == 0||d1(j) == 0)
            continue;
        end
        K(d1(i),d1(j))=K(d1(i),d1(j))+k1(i,j);
    end
end
for i=1:4
    for j=1:4
        if(d2(i)==0||d2(j)==0)
            continue;
        end
        K(d2(i),d2(j))=K(d2(i),d2(j))+k2(i,j);
    end
end
for i=1:4
    for j=1:4
        if(d3(i)==0||d3(j)==0)

```

```

        continue;
    end
    K(d3(i),d3(j))=K(d3(i),d3(j))+k3(i,j);
end
end
for i=1:4
    for j=1:4
        if(d4(i)==0||d4(j)==0)
            continue;
        end
        K(d4(i),d4(j))=K(d4(i),d4(j))+k4(i,j);
    end
end
for i = 1:4
    for j=1:4
        if(d5(i)==0||d5(j)==0)
            continue;
        end
        K(d5(i),d5(j))=K(d5(i),d5(j))+k5(i,j);
    end
end
for i = 1:4
    for j=1:4
        if(d6(i)==0||d6(j)==0)
            continue;
        end
        K(d6(i),d6(j))=K(d6(i),d6(j))+k6(i,j);
    end
end
end
F = [10,-10 0 0]';
U = inv(K)*F;
U1 = U*0;
U2 = U1;
U3 = U1;
U4 = U1;
U5 = U1;
U6 = U1;
U1(3:4) = U(1:2);
U2 = U;
U3(1:2) = U(3:4);
U5(1:2) = U(1:2);
U6(3:4) = U(3:4);
%element 1
%U1 = [26.94 14.42 0 0]';
F1 = T1*k1*U1;
%element 2

```

```

%U2 = [26.94 -14.42 21.36 5.58]';
F2 = T2*k2*U2;
%element 3
%U3 = [21.36 5.58 0 0]';
F3 = T3*k3*U3;
%The nodal displacements of element 4 are zeroes, thus the nodal forces are also
zeroes.
%element 5
%U5 = [26.94 -14.42 0 0]';
F5 = T5*k5*U5;
%element 6
%U6 = [21.36 5.58 0 0]';
F6 = T6*k6*U6;

```

2. Example 2: Plane frame

```

%All rights reserved: Dalian University of Technology
%Matlab implementation of the matrix displacement method
%Solving details:
%-----
clc
E = 1;
%For elements 1 and 3
A1 = 0.5;
l1 = 6;
I1 = 1/24;
%For the horizontal beam
A2 = 0.63;
l2 = 1/12;
l2 = 12;
%Element stiffness matrix of the column
k1 = zeros(6,6);
k1(1,1) = E*A1/l1;
k1(1,4) = -E*A1/l1;
k1(2,2) = 12*E*I1/l1^3;
k1(2,3) = 6*E*I1/l1^2;
k1(2,5) = -12*E*I1/l1^3;
k1(2,6) = 6*E*I1/l1^2;
k1(3,2) = 6*E*I1/l1^2;
k1(3,3) = 4*E*I1/l1;
k1(3,5) = -6*E*I1/l1^2;
k1(3,6) = 2*E*I1/l1;
k1(4,1) = -E*A1/l1;
k1(4,4) = E*A1/l1;
k1(5,2) = -12*E*I1/l1^3;
k1(5,3) = -6*E*I1/l1^2;
k1(5,5) = 12*E*I1/l1^3;

```

```

k1(5,6) = -6*E*I1/l1^2;
k1(6,2) = 6*E*I1/l1^2;
k1(6,3) = 2*E*I1/l1;
k1(6,5) = -6*E*I1/l1^2;
k1(6,6) = 4*E*I1/l1;
%Element stiffness matrix of the horizontal beam
k2 = zeros(6,6);
k2(1,1) = E*A2/l2;
k2(1,4) = -E*A2/l2;
k2(2,2) = 12*E*I2/l2^3;
k2(2,3) = 6*E*I2/l2^2;
k2(2,5) = -12*E*I2/l2^3;
k2(2,6) = 6*E*I2/l2^2;
k2(3,2) = 6*E*I2/l2^2;
k2(3,3) = 4*E*I2/l2;
k2(3,5) = -6*E*I2/l2^2;
k2(3,6) = 2*E*I2/l2;
k2(4,1) = -E*A2/l2;
k2(4,4) = E*A2/l2;
k2(5,2) = -12*E*I2/l2^3;
k2(5,3) = -6*E*I2/l2^2;
k2(5,5) = 12*E*I2/l2^3;
k2(5,6) = -6*E*I2/l2^2;
k2(6,2) = 6*E*I2/l2^2;
k2(6,3) = 2*E*I2/l2;
k2(6,5) = -6*E*I2/l2^2;
k2(6,6) = 4*E*I2/l2;
%For the column, the rotation angle is 90, so the coordinate transformation
matrix is
cosA = 0;
sinA = 1;
T1 = zeros(6,6);
T1(1,1:2) = [cosA sinA];
T1(2,1:2) = [-sinA cosA];
T1(3,3) = 1;
T1(4,4:5) = [cosA sinA];
T1(5,4:5) = [-sinA cosA];
T1(6,6) = 1;
%The element stiffness matrix under the global coordinate system is
K1 = T1'*k1*T1;
K3 = K1;
%For the beam, the rotation angle is 0, so the element stiffness matrix under
%the global coordinate system equals to that under the global coordinate system
K2 = k2;
%Form the global stiffness matrix using direct stiffness method
%The locating vectors of the three elements are

```

```

d1 = [1 2 3 0 0 0];
d2 = [1 2 3 4 5 6];
d3 = [4 5 6 0 0 0];
K = zeros(6,6);
for i = 1:6
    for j=1:6
        if(d1(i)==0||d1(j)==0)
            continue;
        end
        K(d1(i),d1(j))=K(d1(i),d1(j))+K1(i,j);
    end
end
for i=1:6
    for j=1:6
        if(d2(i)==0||d2(j)==0)
            continue;
        end
        K(d2(i),d2(j)) = K(d2(i),d2(j))+K2(i,j);
    end
end
for i = 1:6
    for j = 1:6
        if(d3(i)==0||d3(j)==0)
            continue;
        end
        K(d3(i),d3(j))=K(d3(i),d3(j))+K3(i,j);
    end
end
%The equivalent nodal force vector under the local coordinate system is
q = -1;
f1 = zeros(6,1);
f1(1) = 0;
f1(2) = -q*11*0.5;
f1(3) = -q*11^2/12*(6-8+3);
f1(4) = 0;
f1(5) = -q*11*0.5;
f1(6) = q*11^2/12*(6-8+3);
%Transform the nodal force vector into the global coordinate system
F1 = -T1'*f1;
%Assemble into the global nodal vector
F = zeros(6,1);
F(1:3) = F1(1:3);
%Solve the nodal displacement under the global coordinate system
U = inv(K)*F;
%Solve the nodal force of element 1
U1 = zeros(6,1);

```

```

U1(1:3) = U(1:3);
f1 = T1*(K1*U1-F1);
%Nodal force of element 2
U2 = U;
f2 = K2*U2;
%Nodal force of element 3
U3 = zeros(6,1);
U3(1:3) = U(4:6);
f3 = T1*K3*U3;

```

3. Example 3: Composite structure

```

%All rights reserved: Dalian University of Technology
%Matlab implementation of the matrix displacement method
%Solving details:
%-----
syms E I
A = I/20;
k1_loc = E*I/20*[2 0 0 -2 0 0; 0 0.03 0.3 0 -0.03 0.3; 0 0.3 4 0 -0.3 2; ...
    -2 0 0 2 0 0; 0 -0.03 -0.3 0 0.03 -0.3; 0 0.3 2 0 -0.3 4];
k2_loc = k1_loc;
k3_loc = k1_loc;
k4_loc = E*A*[0.04 0 -0.04 0; 0 0 0 0; -0.04 0 0.04 0; 0 0 0 0];
k5_loc = k4_loc;
T1 = eye(4);
T2 = T1;
T3 = T1;
k1 = k1_loc;
k2 = k2_loc;
k3 = k3_loc;
T4 = [0.8 0.6 0 0; -0.6 0.8 0 0; 0 0 0.8 0.6; 0 0 -0.6 0.8];
k4 = T4*k4_loc*T4;
T5 = [0.8 -0.6 0 0; 0.6 0.8 0 0; 0 0 0.8 -0.6; 0 0 0.6 0.8];
k5 = T5*k5_loc*T5;
%direct stiffness method
%element locating vector
d1 = [0 0 0 1 2 3];
d2 = [1 2 3 4 5 6];
d3 = [4 5 6 0 0 0];
d4 = [0 0 1 2];
d5 = [4 5 0 0];
K = k1*0;
for i = 1:6
    for j = 1:6
        if(d1(i)==0||d1(j)==0)
            continue;
        end
    end
end

```

```

    K(d1(i),d1(j)) = K(d1(i),d1(j))+k1(i,j);
end
end
for i = 1:6
    for j = 1:6
        if(d2(i)==0||d2(j)==0)
            continue;
        end
        K(d2(i),d2(j))=K(d2(i),d2(j))+k2(i,j);
    end
end
for i = 1:6
    for j=1:6
        if(d3(i)==0||d3(j)==0)
            continue;
        end
        K(d3(i),d3(j))=K(d3(i),d3(j))+k3(i,j);
    end
end
for i=1:4
    for j=1:4
        if(d4(i)==0||d4(j)==0)
            continue;
        end
        K(d4(i),d4(j))=K(d4(i),d4(j))+k4(i,j);
    end
end
for i = 1:4
    for j = 1:4
        if(d5(i)==0||d5(j)==0)
            continue;
        end
        K(d5(i),d5(j))=K(d5(i),d5(j))+k5(i,j);
    end
end
F2 = [0 100 333 0 100 -333]';
F = F2;
U = inv(K)*F;

```

4. Example 4: plane frame structure

%All rights reserved: Dalian University of Technology

%Matlab implementation of the matrix displacement method

%Solving details:

%-----

```

k1 = 1.E4*[300 0 0 -300 0 0; 0 12 30 0 -12 30; 0 30 100 0 -30 50;...
    -300 0 0 300 0 0; 0 -12 -30 0 12 -30;0 30 50 0 -30 100];

```



```

k2 = 1.E4*[12 0 -30 -12 0 -30;0 300 0 0 -300 0; -30 0 100 30 0 50;...
-12 0 30 12 0 30; 0 -300 0 0 300 0; -30 0 50 30 0 100];
%direct stiffness method
%element locating vector
d1 = [1 2 3 4 5 6];
d2 = [1 2 3 7 8 9];
K = zeros(9,9);
for i = 1:6
    for j = 1:6
        if(d1(i)==0||d1(j)==0)
            continue;
        end
        K(d1(i),d1(j))=K(d1(i),d1(j))+k1(i,j);
    end
end
for i = 1:6
    for j = 1:6
        if(d2(i)==0||d2(j)==0)
            continue;
        end
        K(d2(i),d2(j))=K(d2(i),d2(j))+k2(i,j);
    end
end
K([4,5,7:9],:)=[];
K(:,[4,5,7:9]) = [];
F1 = [0 12 10 0 12 -10]';
F2 = [4 0 -5 4 0 5]';
F = zeros(9,1);
F(1:6) = F(1:6)+F1;
F(1:3) = F(1:3)+F2(1:3);
F(7:9) = F(7:9)+F2(4:6);
F([4,5,7:9]) = [];
U = inv(K)*F;

```

